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XXXIX. *The Effective Resistance and Inductance of a Concentric Main, and Methods of Computing the Ber and Bei and Allied Functions.* By ALEXANDER RUSSELL, *M.A., D.Sc.**

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I. INTRODUCTION.

IN connexion with the investigation of certain phenomena which occur when alternating currents of high frequency flow in thick wires, a knowledge of how the effective resistances and inductances of these wires vary with the frequency is most important. It is of little use to have "standard" inductances in high frequency circuits when we do not know how their values alter with the frequency, and therefore also with the wave shape of the alternating currents. The "measurement" of inductances by means of alternating currents of unknown wave shape often leads to waste of time. Even

* Read January 22, 1909.

when the wave shape is similar to a sine curve, yet if we do not know how the inductance will vary with the frequency the results have only a very limited application.

It would obviously be extremely useful to have formulæ which would take into account the appreciable variation of the density of the current which occurs over the cross section of the wires with high frequency currents. Even to fix inferior and superior limits to the possible values of the inductance would be a great help in many cases. Unfortunately the mathematical difficulties in the way of arriving at a solution in the case of a helical coil are very great. The author, therefore, has made a study of the simplest problem of all, namely that of a concentric main, as a preliminary to attacking the more difficult problems. A study of this problem is also of importance at the present time * in connexion with the discussion that is taking place amongst electrical engineers as to the magnitude of the skin losses in power transmission cables.

The problem was first discussed by Maxwell †. He obtains a few of the terms of a series by means of which the effective resistance of the inner conductor can be computed at low frequencies. Apparently, however, he did not fully appreciate the importance of the results given by the formula. In May ‡ 1884, Oliver Heaviside discussed the "throttling" effect in a core, that is, the increased resistance, the reduced inductance, and the tendency to surface concentration. He uses two functions M and N in his solution, which Kelvin subsequently called the ber and bei functions. In January § 1885, he described clearly the true nature of the current flow in a wire, laying particular stress on the initial surface effects and subsequent penetration. Lord Rayleigh ‖, adopting

* See the report of the evidence on the London Electric Power Bills given before Sir Luke White's Committee in the House of Lords (Nov. 1908).

† 'Electricity and Magnetism,' vol. ii. § 690.

‡ 'The Electrician,' p. 583, May 3, 1884, or 'Electrical Papers,' vol. i. p. 353.

§ 'The Electrician,' Jan. 3, 1885, or 'Electrical Papers,' vol. i. p. 429.

‖ Phil. Mag. xxii. pp. 381–394 (1886), or 'Scientific Papers,' vol. ii. p. 86.

Maxwell's method, next discussed the problem and gave a formula for the effective resistance of the inner conductor at very high frequencies. Oliver Heaviside * subsequently made most important contributions to our knowledge of the subject. He was the first, for example, to give the approximate formula† for the effective resistance of a hollow cylindrical conductor carrying a low frequency current. This formula is a particular case of the general formula given in this paper. He also gives general descriptions of how the current-density varies in the conductors. He omits so many steps, however, in some places that it is very laborious to follow his reasoning. Three years afterwards Lord Kelvin ‡ gave a practical solution for the effective resistance of a solid inner conductor. It is virtually the same as that given by Heaviside. He gave a table, which we shall examine later, of the numerical values in important practical cases. Sir Joseph Thomson § also gives practical formulæ for the effective resistance and inductance of a concentric main having a solid inner conductor when traversed by very high frequency currents.

The complete solution given in this paper is obtained from elementary electrical considerations, a knowledge of Ohm's law and of Faraday's law of induction being all that is assumed. The author proves the mathematical formulæ at length, as most of them are new and some of them will be helpful in other physical problems. It will also enable any slips he may have made in the algebraical work to be readily detected and easily rectified. He shows, however, that from his solutions all the previous solutions can be readily deduced, and as most of them are complex functions of the electrical and geometrical data of the main, the errors, if any, must be very minor ones.

* 'Electrical Papers,' vol. ii. p. 64 *et seq.*

† *L. c. ante*, p. 192, formula (72).

‡ Journ. of the Inst. of El. Eng. vol. xviii. p. 4 (1889) or Math. and Phys. Papers, vol. iii. p. 491.

§ 'Recent Researches,' p. 295.

II. Mathematical Formulæ.

1. *The Differential Equation.*

The differential equation * to which our problem leads is

$$\frac{\partial^2 i}{\partial r^2}+\frac{1}{r}\frac{\partial i}{\partial r}=\frac{m^2}{\omega}\frac{\partial i}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where m and ω are constants and i is a periodic function. If we assume that i varies according to the harmonic law, and that its frequency is $\omega/2\pi$, we may write $i=u\epsilon^{\omega t\iota}$, where u is a function of r but not of t, and ι stands for $\sqrt{-1}$.

The equation now becomes

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-m^2\iota u=0. \quad . \quad . \quad . \quad . \quad (B)$$

The solution of this equation is known † to be

$$u=A\,.\,I_0(mr\sqrt{\iota})+B\,.\,K_0(mr\sqrt{\iota}),$$

where A and B are constants, and $I_0(x)$ and $K_0(x)$ can be computed by means of the following series :—

When x is small,

$$I_0(x)=1+\frac{x^2}{2^2}+\frac{x^4}{2^2\,.\,4^2}+\ldots\ldots \quad . \quad . \quad . \quad (1)$$

and

$$K_0(x)=\alpha\,.\,I_0(x)-\log x\,.\,I_0(x)+\frac{x^2}{2^2}+\left(1+\frac{1}{2}\right)\frac{x^4}{2^2\,.\,4^2}+\ldots \quad (2)$$

where $\alpha=\log 2-\gamma$ and γ is Euler's constant, and so

$$\alpha=0{\cdot}1159315.$$

When x is large,

$$I_0(x)=\frac{\epsilon^x}{\sqrt{2\pi x}}\left\{1+\frac{1^2}{8x}+\frac{1^2\,.\,3^2}{\underline{|2}(8x)^2}+\ldots\ldots\right\}, \quad . \quad . \quad (3)$$

and

$$K_0(x)=\sqrt{\frac{\pi}{2x}}\,.\,\epsilon^{-x}\left\{1-\frac{1^2}{8x}+\frac{1^2\,.\,3^2}{\underline{|2}(8x)^2}-\ldots\ldots\right\}. \quad (4)$$

The values of the series (2) and (4), for many values of x,

* This equation was first discussed by Joseph Fourier [*Mémoires de l'Académie*, Tome iv. (for the year 1819)].

† See Gray and Matthews, 'Bessel's Functions.'

have been computed to a high degree of accuracy by W. S. Aldis *, and tables of $I_0(x)$ from 0 to 5·1 are given in the British Assoc. Reports, 1896, the interval of the argument being 0·001.

2. *Kelvin's* ber *and* bei *functions.*

Kelvin † showed that the effective resistance of the inner conductor of a concentric main may be conveniently expressed in terms of two functions which he called the ber and the bei functions ‡. He published tables of the values of these functions for a few values of the argument, and from these tables he computed the numerical value of the effective resistance in various cases.

We may define Kelvin's functions by means of the equation

$$I_0(mr\sqrt{\iota}) = \mathrm{ber}\, mr + \iota\, \mathrm{bei}\, mr.$$

From (1) we deduce at once that

$$\mathrm{ber}\, mr = 1 - \frac{m^4r^4}{2^2 \cdot 4^2} + \frac{m^8r^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \dots \quad . \quad . \quad . \quad (5)$$

and

$$\mathrm{bei}\, mr = \frac{m^2r^2}{2^2} - \frac{m^6r^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots, \quad . \quad . \quad . \quad . \quad (6)$$

which are the definitions of them given by Kelvin.

In Mascart and Joubert's *l'Electricité et le Magnetisme*, vol. i. p. 718, several of the values of the functions given in Kelvin's paper have been recomputed and certain corrections made. As the author uses these functions in the solutions given below, it was necessary therefore to recheck the calculations. He at first attempted to do this by direct calculation, but the work proved so laborious that he was led to devise shorter methods of calculating the functions. He found that this was easy and that comparatively simple formulæ can be obtained for them in those cases where the direct computation by (5) and (6) would be laborious.

* Proc. Roy. Soc. vol. lxiv. p. 203.

† *L. c. ante.*

‡ Noticing that in Heaviside's notation, ber=M and bei=N, Kelvin's formula follows at once from the formula (36) given in Heaviside's 'Electrical Papers,' vol. ii. p. 183.

The formulæ given below can also be usefully employed in simplifying the formulæ ordinarily given for computing the eddy-current losses * in a metallic cylinder, the inductance and resistance of two parallel cylindrical conductors †, the impedance of a solenoid with a cylindrical metal core ‡, &c.

3. *Approximate formulæ for the* ber *and* bei *functions.*

When the argument is small the functions can be readily computed from the formulæ (5) and (6). These functions generally occur associated together in one or other of the following ways :—

$$X(x) = \mathrm{ber}^2 x + \mathrm{bei}^2 x, \qquad Y(x) = \mathrm{ber}'^2 x + \mathrm{bei}'^2 x,$$

$$Z(x) = \mathrm{ber}\, x\, \mathrm{ber}'\, x + \mathrm{bei}\, x\, \mathrm{bei}'\, x,$$

and $$W(x) = \mathrm{ber}\, x\, \mathrm{bei}'\, x - \mathrm{bei}\, x\, \mathrm{ber}'\, x.$$

In these definitions of X, Y, Z and W, ber′ x and bei′ x stand for the differential coefficients of ber x and bei x with respect to x. The combinations Y/X, Z/X, W/X, Z/Y, and W/Y also occur in this and allied problems, and so we shall give formulæ for these functions as well.

By squaring the series for ber x and bei x and adding them together, we get

$$X = 1 + \frac{1}{2}\left(\frac{x}{2}\right)^4 + \frac{1}{4\underline{|4}}\left(\frac{x}{2}\right)^8 + \frac{1}{6^2\underline{|6}}\left(\frac{x}{2}\right)^{12} + \frac{1}{8^2\underline{|9}}\left(\frac{x}{2}\right)^{16}, \quad (7)$$

approximately.

When x is not greater than 4 this formula may be used. For instance, when x is 4, (7) gives $X(4) = 11{\cdot}8275$. From the tables given in Gray and Matthews' 'Bessel's Functions' where the values of ber x and bei x are tabulated for values of x up to 6, the interval of the argument being 0·2, we find that ber $4 = -2{\cdot}56342$ and bei $4 = 2{\cdot}29269$, and thus $X(4) = \mathrm{ber}^2 4 + \mathrm{bei}^2 4 = 11{\cdot}8275$.

* A. Russell, 'Alternating Currents,' vol. i. p. 374.

† J. W. Nicholson, Phil. Mag. [6] xvii. p. 255, 1909.

‡ O. Heaviside, 'Electrical Papers,' vols. i. and ii., or R. T. Wells, Phys. Rev. xxvi. p. 357 (1908).

Similarly we get the approximate formulæ :—

$$Y=\frac{x^2}{4}\left\{1+\frac{3}{(\underline{|3})^2}\left(\frac{x}{2}\right)^4+\frac{10}{(\underline{|5})^2}\left(\frac{x}{2}\right)^8+\frac{35}{(\underline{|7})^2}\left(\frac{x}{2}\right)^{12}\right\}, \quad . \quad . \quad (8)$$

$$Z=\frac{x^3}{16}\left\{1+\frac{1}{\underline{|4}}\left(\frac{x}{2}\right)^4+\frac{1}{6\,\underline{|6}}\left(\frac{x}{2}\right)^8+\frac{1}{4\,.\,12^2\,.\,\underline{|7}}\left(\frac{x}{2}\right)^{12}\right\}, \quad (9)$$

and

$$W=\frac{x}{2}\left\{1+\frac{6}{(\underline{|3})^2}\left(\frac{x}{2}\right)^4+\frac{30}{(\underline{|5})^2}\left(\frac{x}{2}\right)^8+\frac{140}{(\underline{|7})^2}\left(\frac{x}{2}\right)^{12}\right\}. \quad . \quad . \quad (10)$$

When x is not greater than 2 the following formulæ can be employed :—

$$\frac{Y}{X}=\frac{x^2}{4}\left\{1-\frac{5}{12}\left(\frac{x}{2}\right)^4+\frac{143}{720}\left(\frac{x}{2}\right)^8-\frac{7661}{4^2\,\underline{|7}}\left(\frac{x}{2}\right)^{12}\right\}, \quad . \quad . \quad . \quad (11)$$

$$\frac{Z}{X}=\frac{x^3}{16}\left\{1-\frac{11}{24}\left(\frac{x}{2}\right)^4+\frac{473}{3\,\underline{|6}}\left(\frac{x}{2}\right)^8-\frac{304107}{4\,.\,12^2\,\underline{|7}}\left(\frac{x}{2}\right)^{12}\right\}, \quad . \quad (12)$$

$$\frac{W}{X}=\frac{x}{2}\left\{1-\frac{1}{3}\left(\frac{x}{2}\right)^4+\frac{19}{\underline{|5}}\left(\frac{x}{2}\right)^8-\frac{687}{7\,.\,6^4}\left(\frac{x}{2}\right)^{12}\right\}, \quad . \quad . \quad . \quad (13)$$

$$\frac{Z}{Y}=\frac{x}{4}\left\{1-\frac{1}{24}\left(\frac{x}{2}\right)^4+\frac{13}{4320}\left(\frac{x}{2}\right)^8-\frac{647}{12^2\,.\,360\,.\,56}\left(\frac{x}{2}\right)^{12}\right\}, \quad (14)$$

and

$$\frac{W}{Y}=\frac{2}{x}\left\{1+\frac{1}{12}\left(\frac{x}{2}\right)^4-\frac{1}{180}\left(\frac{x}{2}\right)^8+\frac{11}{12\,.\,28\,.\,30}\left(\frac{x}{2}\right)^{12}\right\}. \quad (15)$$

Formula (15) agrees with that found by Lord Rayleigh * by another method. Heaviside † gives $73/(12^2\,.\,28\,.\,80)$, that is $657/(12^2\,.\,360\,.\,56)$ as the coefficient of $(x/2)^{12}$ in (14).

Formulæ (7) to (10) should not be used if x is greater than 4, and formulæ (11) to (15) should not be used if x is greater than 2. In practice, therefore, they have only a limited use.

We shall now find approximate formulæ for ber x, bei x, X, Y, Z, and W, which can be used with sufficient accuracy for practical purposes when x is not less than 5, and can

* *L. c. ante.*

† 'Electrical Papers,' vol. ii. p. 64. [Dr. Heaviside has written to me that he discovered this slip in 1894.]

always be used for computing with a maximum inaccuracy of less than 1 in 10,000, when x is not less than 10, that is in those cases where the labour involved in the direct computation of the series becomes practically prohibitive.

If y denote $\operatorname{ber} x + \iota \operatorname{bei} x$ we see from the definition we gave of these functions that

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{x}\frac{\partial y}{\partial x} = \iota y. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Putting $y = A'\epsilon^{\theta}/\sqrt{x}$, where A' is a constant, we get

$$\frac{\partial^2 \theta}{\partial x^2} + \left(\frac{\partial \theta}{\partial x}\right)^2 - \iota + \frac{1}{4x^2} = 0. \quad . \quad . \quad . \quad (17)$$

When x is large,

$$\theta = x\sqrt{\iota} + a_0 + b_0\iota$$

is obviously an approximate solution of (17), where a_0 and b_0 are constants. Let us assume therefore that

$$\theta = x\sqrt{\iota} + a_0 + b_0\iota + a_1/x + a_2/x^2 + \ldots$$

is a solution of (17). Substituting this value for θ, equating the coefficients of x^{-2}, x^{-3}, ... to zero, and noticing that $\sqrt{\iota} = 1/\sqrt{2} + (1/\sqrt{2})\iota$, and $1/\sqrt{\iota} = 1/\sqrt{2} - (1/\sqrt{2})\iota$, we find that

$$a_1 = \frac{1}{8\sqrt{2}} - \frac{\iota}{8\sqrt{2}}, \qquad a_2 = -\frac{\iota}{16},$$

$$a_3 = -\frac{25}{384\sqrt{2}} - \frac{25\iota}{384\sqrt{2}}, \qquad a_4 = -\frac{13}{128},$$

&c.

Hence we may write

$$y\sqrt{x} = A'\epsilon^{\alpha + a_0 + \beta\iota}$$

$$= A\epsilon^{\alpha}\cos\beta + \iota A\epsilon^{\alpha}\sin\beta,$$

where

$$\alpha = \frac{x}{\sqrt{2}} + \frac{1}{8\sqrt{2}x} - \frac{25}{384\sqrt{2}x^3} - \frac{13}{128x^4} - \ldots \quad . \quad (18)$$

and

$$\beta = \frac{x}{\sqrt{2}} + b_0 - \frac{1}{8\sqrt{2}x} - \frac{1}{16x^2} - \frac{25}{384\sqrt{2}x^3} + \ldots \quad . \quad (19)$$

Hence, since $y = \text{ber}\, x + \iota\, \text{bei}\, x$, we have

$\text{ber}\, x = (A/\sqrt{x})\, \epsilon^{a} \cos\beta$, and $\text{bei}\, x = (A/\sqrt{x})\epsilon^{a} \sin\beta$.

To determine the values of A and b_0 we notice that

$$\text{ber}\, x + \iota\, \text{bei}\, x = I_0(x\sqrt{\iota}),$$

and thus, from equation (3), we see that A is $1/\sqrt{2\pi}$, and b_0 is $-\pi/8$.

We thus find that when x is large

$$\text{ber}\, x = \frac{\epsilon^{a}}{\sqrt{2\pi x}} \cos\beta, \quad \ldots \quad (20)$$

and

$$\text{bei}\, x = \frac{\epsilon^{a}}{\sqrt{2\pi x}} \sin\beta, \quad \ldots \quad (21)$$

where α and β are given by (18) and (19), and b_0 is $-\pi/8$.

The series for α and β are semi-convergent and a rigorous mathematical justification of (20) and (21) is difficult. It is easy, however, to verify that, if we only include the terms of the series given above, (20) and (21) give the values of the functions with great accuracy when x is greater than 5. The values of α and β are easily computed by the formulæ

$$\alpha = 0\cdot707105\, x + 0\cdot08839/x - 0\cdot046/x^3, \quad \ldots \quad (22)$$

and

$$\beta = 0\cdot707105\, x - 0\cdot39270 - 0\cdot08839/x - 0\cdot0625/x^2 - 0\cdot046/x^3. \quad (23)$$

Differentiating (20) and (21) we find that

$$\text{ber}'\, x = \left(\frac{1}{\sqrt{2}} - \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2}\right) \text{ber}\, x - \left(\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} + \frac{1}{8x^3}\right) \text{bei}\, x, . \quad (24)$$

and

$$\text{bei}'\, x = \left(\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} + \frac{1}{8x^3}\right) \text{ber}\, x + \left(\frac{1}{\sqrt{2}} - \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2}\right) \text{bei}\, x. \quad (25)$$

Squaring equations (20) and (21), and adding, we get

$$X = \frac{\epsilon^{2a}}{2\pi x}, \quad \ldots \quad (26)$$

When the value of x is not less than 7, the inaccuracy of the formula

$$X = \frac{\epsilon^{x\sqrt{2} + 1/4\sqrt{2}x}}{2\pi x}$$

is less than 1 in 10,000.

Heaviside * has given the formula $X=\epsilon^{x\sqrt{2}}/2\pi x$, but in order to get a four-figure accuracy with this formula x would have to be greater than 1500.

In a similar manner, we find that

$$Y=X\left(1-\frac{1}{x\sqrt{2}}+\frac{1}{4x^2}+\frac{3}{8\sqrt{2x^3}}\right), \quad . \quad . \quad (27)$$

$$Z=X\left(\frac{1}{\sqrt{2}}-\frac{1}{2x}-\frac{1}{8\sqrt{2x^2}}\right), \quad . \quad . \quad . \quad . \quad (28)$$

$$W=X\left(\frac{1}{\sqrt{2}}+\frac{1}{8\sqrt{2x^2}}+\frac{1}{8x^3}\right). \quad . \quad . \quad . \quad . \quad (29)$$

These formulæ also give the ratios Y/X, Z/X, and W/X. They correspond to (11), (12), and (13), and give a four-figure accuracy when x is not less than 8.

By the binomial theorem, we also readily deduce the following formulæ corresponding to (14) and (15) :—

$$\frac{Z}{Y}=\frac{1}{\sqrt{2}}-\frac{3}{8\sqrt{2x^2}}-\frac{3}{8x^3} \quad . \quad . \quad . \quad (30)$$

and

$$\frac{W}{Y}=\frac{1}{\sqrt{2}}+\frac{1}{2x}+\frac{3}{8\sqrt{2x^2}}. \quad . \quad . \quad . \quad . \quad (31)$$

To test these formulæ let us take the low value of 6 for x. We find from (20) and (21) that

$$\text{ber } 6=-8{\cdot}858 \quad \text{and} \quad \text{bei } 6=-7{\cdot}335.$$

These results are in exact agreement with their values found by direct computation from the series given in (5) and (6). The accuracy of the formulæ rapidly increases as x increases.

In the following table the numbers obtained by substituting 10 for x in formulæ (20), (21), (24), and (25) are compared with the numbers given by Kelvin † and by Mascart and Joubert †.

* 'Electrical Papers,' vol. ii. p. 184.

† *L. c. ante.*

	ber 10.	bei 10.	ber′ 10.	bei′ 10.
Kelvin	138·8405	56·3704	51·373	135·23
Mascart & Joubert ...	138·840	56·370	51·207	135·31
Formulæ	138·840	56·368	51·202	135·29

Before the author worked out the approximate formulæ, he had verified Mascart and Joubert's corrections as far as four figures by direct calculation from the series given in (5) and (6) and the series obtained by differentiating them. As the corrections to Kelvin's table when the argument is 15 and when it is 20 are large, it will be interesting to calculate these values by our formulæ.

	ber 15.	bei 15.	ber′ 15.	bei′ 15.
Kelvin	−2969·79	−2952·33	86·648	−4089·2
Mascart & Joubert ...	−2967·26	−2952·72	91·061	−4088·5
Formulæ	−2967·26	−2952·66	91·010	−4087·7

	ber 20.	bei 20.	ber′ 20.	bei′ 20.
Kelvin	47583·7	11500·8	24325·1	41491·5
Mascart & Joubert ...	47489·2	114774·4	−48802·8	111853
Formulæ	47491·6	114770·1	−48797·9	111853

In evaluating the formulæ we have only used 7-figure logarithmic tables. To compute ber′ 15 correctly we ought to have calculated ber 15 and bei 15 to eight significant figures, as ber 15 and bei 15 are nearly equal to one another and from (24) we see that ber′ 15 is nearly equal to their difference. It will be seen that the accuracy of Mascart and Joubert's corrections is satisfactory. The values, however, they give for the functions when the argument is 30 are not correct.

	ber 30.	bei 30.	ber′ 30.	bei′ 30.
Mascart & Joubert ...	-4544×10^4	11024×10^4	-10933×10^4	4383×10^4
Formulæ	-4611×10^4	10995×10^4	-10959×10^4	4330×10^4

Hence the ordinary tables need revision. A more useful set of tables, however, might be constructed of the functions X, Y, Z, W, Y/X, Z/X, W/X, Z/Y, and W/Y. The ber and bei functions never occur alone in any of the practical formulæ using these functions, with which the author is acquainted.

For instance, Heaviside * and Kelvin * have proved that the ratio of the effective resistance of the inner core of a concentric main with high frequency currents to its resistance with direct currents can be written down almost at once, when the ratio of W to Y is known. It was in fact in order to find the value of this ratio that Kelvin had a table of ber and bei functions computed.

The values of W/Y given in Kelvin's paper and found by (15) and also by direct computation are compared in the following table :—

x.	Values given in Kelvin's paper.	Values computed by (15).	True Values.
0·5	4·0000	4·0013	4·0013
1·0	2·00014	2·0104	2·0104
1·5	1·3678	1·3678	1·3678
2·0	1·0805	1·0782	1·0782

In Kelvin's table bei′1 is given as 0·4974, but its true value is 0·4974. Making this correction and using Kelvin's figures, we get 2·0104 for the value of W/Y when x is 1.

Including the next term in the expansion of W/Y given in (31), we find that

$$\frac{W}{Y} = 0{\cdot}7071 + \frac{1}{2x} + \frac{0{\cdot}265}{x^2} - \frac{0{\cdot}35}{x^4}. \quad . \quad . \quad (32)$$

* *L. c. ante.*

The values of W Y given in the first column of the following table have been computed by this formula.

x.	Kelvin.	Mascart & Joubert.	Formula (32).
5	0·8172	......	0·8172
5·5	0·8069	......	0·8064
6	0·7979	......	0·7976
8	0·7739	......	0·7737
10	0·7588	0·7596	0·7597
15	0·7431	0·7416	0·7416
20	0·7325	0·7328	0·7326
30	......	0·7251	0·7241
40	......	0·7196	0·7198
50	......	0·7172	0·7172
∞	0·7071	......	0·7071

As the approximate formulæ given above are very simple, it will be seen that tables need only be constructed for values of x lying between 2 and 5 or 6, although to have tables of other values would doubtless be a great convenience to those who have to use the formulæ.

4. *Particular solution.*

We have seen that a solution of the equation

$$\frac{\partial^2 i}{\partial r^2}+\frac{1}{r}\frac{\partial i}{\partial r}=\frac{m^2}{\omega}\frac{\partial i}{\partial t}$$

is
$$i = I_0(mr\sqrt{\iota})\,e^{\iota\omega t}$$
$$= (\mathrm{ber}\,mr+\iota\,\mathrm{bei}\,mr)(\cos\omega t+\iota\sin\omega t).$$

Hence, since both the real and imaginary parts of this solution must satisfy the differential equation, we see that a particular solution may be written in either of the following forms:

$$i = (\mathrm{A}\,\mathrm{ber}\,mr+\mathrm{B}\,\mathrm{bei}\,mr)\cos\omega t$$
$$+(-\mathrm{A}\,\mathrm{bei}\,mr+\mathrm{B}\,\mathrm{ber}\,mr)\sin\omega t, \quad . \quad . \quad (33)$$

or
$$i = (\mathrm{A}^2+\mathrm{B}^2)^{\frac{1}{2}}(\mathrm{ber}^2\,mr+\mathrm{bei}^2\,mr)^{\frac{1}{2}}\cos(\omega t-\epsilon), \quad . \quad . \quad (34)$$

where A and B are constants, and

$$\tan\epsilon = (-\mathrm{A}\,\mathrm{bei}\,mr+\mathrm{B}\,\mathrm{ber}\,mr)/(\mathrm{A}\,\mathrm{ber}\,mr+\mathrm{B}\,\mathrm{bei}\,mr).$$

For a solid core and an infinitely thin return conductor of infinite conductivity this solution suffices, and is the one given by Kelvin. When, however, the core is hollow * or when we wish to take into account the effects of the return conductor, the complete solution has to be found, as the above solution cannot be made to satisfy all the boundary conditions.

5. *The* ker *and* kei *functions.*

Imitating Kelvin we shall write the second type of solution of (B), namely $K_0(mr\sqrt{\iota})$, in the form $\ker mr + \iota \operatorname{kei} mr$. Substituting $x\sqrt{\iota}$ for x in equation (2) and equating the coefficients of the real and imaginary terms on the sides of the equation, we find that

$$\ker x = (\alpha - \log x)\operatorname{ber} x + (\pi/4)\operatorname{bei} x - (1+\tfrac{1}{2})\frac{x^4}{2^2.4^2} + (1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})\frac{x^8}{2^2.4^2.6^2.8^2} - \ldots, \qquad (35)$$

and

$$\operatorname{kei} x = (\alpha - \log x)\operatorname{bei} x - (\pi/4)\operatorname{ber} x + \frac{x^2}{2^2} - (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{x^6}{2^2.4^2.6^2} + \ldots, \qquad (36)$$

where $\alpha = 0{\cdot}1159315\ldots$.

6. *Approximate formulæ for the* ker *and* kei *functions.*

When x is small, we may use the following approximate formulæ, which can easily be proved by (5) and (6) :—

$$\ker mx = \alpha - \log mx + \frac{\pi}{4}\cdot\frac{m^2x^2}{4} - (\alpha + \tfrac{3}{2} - \log mx)\frac{m^4x^4}{64} - \frac{\pi}{4}\cdot\frac{m^6x^6}{2^2.4^2.6^2}, \qquad (37)$$

$$\operatorname{kei} mx = -\frac{\pi}{4} + (\alpha + 1 - \log mx)\frac{m^2x^2}{4} + \frac{\pi}{4}\cdot\frac{m^4x^4}{64} - (\alpha + \tfrac{11}{6} - \log mx)\frac{m^6x^6}{2^2.4^2.6^2},$$

$$\ker' mx = -\frac{1}{mx} + \frac{\pi}{4}\cdot\frac{mx}{2} - (\alpha + \tfrac{5}{4} - \log mx)\frac{m^3x^3}{16} - \frac{\pi}{4}\cdot\frac{m^5x^5}{2^2.4^2.6}, \qquad (39)$$

and

$$\operatorname{kei}' mx = (\alpha + \tfrac{1}{2} - \log mx)\frac{mx}{2} + \frac{\pi}{4}\cdot\frac{m^3x^3}{16} - (\alpha + \tfrac{5}{3} - \log mx)\frac{m^5x^5}{2^2.4^2.6}. \qquad (40)$$

* The solution given in Mascart & Joubert, vol. i. p. 719 (1897), for the effective resistance of a hollow inner core is incorrect.

When mx is small we see that

$$\operatorname{ker} mx = \alpha - \log mx \text{ and } \operatorname{kei} mx = -\pi/4, \text{ approximately;}$$

and hence, when mx is very small, we may write

$$\operatorname{ker}^2 mx + \operatorname{kei}^2 mx = (\log mx)^2. \quad . \quad . \quad . \quad (41)$$

When x is not small we get suitable formulæ for calculating these functions by noticing that

$$\theta = -x\sqrt{\iota} + a_0' + b_0'\sqrt{\iota}$$

is also an approximate solution of (17). Hence, finding a series in descending powers of x for θ, and determining the constants by (4), we find that

$$\operatorname{ker} x = \sqrt{\frac{\pi}{2x}}\epsilon^{\alpha'} \cos \beta', \quad . \quad . \quad . \quad . \quad (42)$$

and

$$\operatorname{kei} x = \sqrt{\frac{\pi}{2x}}\epsilon^{\alpha'} \sin \beta', \quad . \quad . \quad . \quad . \quad (43)$$

where

$$\alpha' = -\frac{x}{\sqrt{2}} - \frac{1}{8\sqrt{2}x} + \frac{25}{384\sqrt{2}x^3} - \frac{13}{128\,x^4} + \ldots \quad . \quad (44)$$

and
$$\beta' = -\frac{x}{\sqrt{2}} - \frac{\pi}{8} + \frac{1}{8\sqrt{2}x} - \frac{1}{16\,x^2} + \frac{25}{384\sqrt{2}x^3} - \ldots. \quad (45)$$

It will be seen that α' and β' can be deduced from the formulæ for α and β, (18) and (19), by merely changing the sign of x in the latter. In making calculations it is best to use the formulæ obtained by writing $-x$ for x on the right-hand side of the equations (22) and (23).

By differentiating (42) and (43) we find that

$$\operatorname{ker}' x = -\operatorname{ker} x\left\{\frac{1}{\sqrt{2}} + \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2}\right\}$$

$$+\operatorname{kei} x\left\{\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} - \frac{1}{8x^3}\right\}, \quad . \quad . \quad . \quad (46)$$

and
$$\operatorname{kei}' x = -\operatorname{ker} x\left\{\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} - \frac{1}{8x^3}\right\}$$

$$-\operatorname{kei} x\left\{\frac{1}{\sqrt{2}} + \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2}\right\}. \quad . \quad . \quad . \quad (47)$$

We also have

$$X_1(x) = \ker^2 x + \mathrm{kei}^2 x = \frac{\pi\epsilon^{2a'}}{2x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and

$$Y_1(x) = \ker'^2 x + \mathrm{kei}'^2 x = X_1(x)\left\{1+\frac{1}{x\sqrt{2}}+\frac{1}{4x^2}-\frac{3}{8\sqrt{2}x^3}\right\} . \quad (49)$$

When mx is very large we shall use the formulæ

$$\ker mx = \sqrt{\frac{\pi}{2mx}}\,\epsilon^{-\frac{mx}{\sqrt{2}}}\cos\left(\frac{mx}{\sqrt{2}}+\frac{\pi}{8}\right), \quad . \quad . \quad . \quad (50)$$

$$\mathrm{kei}\, mx = -\sqrt{\frac{\pi}{2mx}}\,\epsilon^{-\frac{mx}{\sqrt{2}}}\sin\left(\frac{mx}{\sqrt{2}}+\frac{\pi}{8}\right), \quad . \quad . \quad (51)$$

$$\ker' mx = -\sqrt{\frac{\pi}{2mx}}\,\epsilon^{-\frac{mx}{\sqrt{2}}}\cos\left(\frac{mx}{\sqrt{2}}-\frac{\pi}{8}\right), \quad . \quad . \quad (52)$$

and

$$\mathrm{kei}' mx = \sqrt{\frac{\pi}{2mx}}\,\epsilon^{-\frac{mx}{\sqrt{2}}}\sin\left(\frac{mx}{\sqrt{2}}-\frac{\pi}{8}\right). \quad . \quad . \quad (53)$$

7. *Formulæ containing both functions.*

We require the following formulæ, also, in our solutions. When mc is small,

$$S_c = \mathrm{ber}'\, mc \ker'\, mc + \mathrm{bei}'\, mc\, \mathrm{kei}'\, mc$$

$$= \frac{m^2c^2}{4}\left(\alpha+\tfrac{3}{4}-\log mc\right)$$

$$+\frac{m^6c^6}{12\,.\,64}(\alpha-\log mc)+\frac{37m^6c^6}{2^2.\,4^2.\,6^2.\,8}, \quad . \quad (54)$$

and

$$T_c = \mathrm{bei}'\, mc \ker'\, mc - \mathrm{ber}'\, mc\, \mathrm{kei}'\, mc$$

$$= -\frac{1}{2}+\frac{\pi}{4}\cdot\frac{m^2c^2}{4}-\frac{m^4c^4}{48}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

When mc is very great, we may write

$$S_c = -\frac{\cos mc\sqrt{2}}{2mc}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

and

$$T_c = -\frac{\sin mc\sqrt{2}}{2mc}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

8. *The complete solution.*

On the assumption that i follows the harmonic law, we see that the complete solution of the equation (A) is

$$i = (\mathrm{A}\,\mathrm{ber}\,mr + \mathrm{B}\,\mathrm{bei}\,mr + \mathrm{C}\,\mathrm{ker}\,mr + \mathrm{D}\,\mathrm{kei}\,mr)\cos\omega t$$
$$+ (-\mathrm{A}\,\mathrm{bei}\,mr + \mathrm{B}\,\mathrm{ber}\,mr - \mathrm{C}\,\mathrm{kei}\,mr + \mathrm{D}\,\mathrm{ker}\,mr)\sin\omega t, \quad (58)$$

where A, B, C, and D are constants.

III. The Formulæ for the effective Resistance and Inductance of a Concentric Main with a Solid Inner Conductor.

In order to simplify the problem, we shall first suppose that the inner conductor is a solid metal cylinder of radius a, and that the outer conductor is a coaxial hollow cylinder of inner and outer radii b and c respectively. Let μ be the value of the permeability of the metals forming the conductors, and let μ' be the permeability of the insulating material separating them. Let ρ be the volume resistivity of the conducting metal. We shall suppose that μ, μ', and ρ are constants and, for the present, that both the capacity and leakage currents in the dielectric can be neglected. We can assume, therefore, that the flow of current in the conductors is parallel to their common axis, and hence, that the equipotential surfaces in each conductor are planes perpendicular to this axis.

Let us now consider the current in a cylindrical tube of unit length in the inner conductor, whose inner and outer radii are r and $r+dr$ respectively. If e_1 be the potential difference between the ends of this tube, the equation to determine the current-density i in it is, by Ohm's law and Faraday's law,

$$e_1 = (\rho/2\pi r\partial r)(i\,.\,2\pi r\partial r) + \partial\phi/\partial t$$
$$= \rho i + \partial\phi/\partial t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where ϕ is the number of magnetic lines linked with the current in this cylindrical tube.

By hypothesis, the equipotential surfaces in the inner conductor are planes perpendicular to the axis. Hence e_1 is

independent of the value of r, and thus

$$0 = \rho \frac{\partial i}{\partial r} + \frac{\partial}{\partial t} \frac{\partial \phi}{\partial r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

From the symmetry of a concentric main, we see that the intensity of the current is the same at all points equidistant from the axis. Hence, since the magnetic force outside an infinite cylindrical tube, carrying a current flowing parallel to its axis, is the same as if all the current were concentrated at this axis, we have, at all points of the inner conductor,

$$\phi = \mu \int_r^a \frac{2\mathrm{I}_x}{x} \partial x + 2\mu' \mathrm{I} \log \frac{b}{a} + \mu \int_b^c \frac{2(\mathrm{I} - \mathrm{I}'_x)}{x} \partial x, \quad (61)$$

where I_x is the algebraical sum of the currents flowing through the cross section of a coaxial cylinder whose radius is x, I is the total current flowing in the inner conductor, and I'_x is the sum of the currents flowing in the tube of the outer conductor whose inner radius is b and outer radius is x.

By differentiating ϕ with respect to r, we get

$$\frac{\partial \phi}{\partial r} = -\frac{2\mu}{r} \int_0^r 2\pi i r \partial r,$$

and hence, by (60),

$$\rho \frac{\partial i}{\partial r} = \frac{4\pi\mu}{r} \int_0^r r \frac{\partial i}{\partial t} \partial r, \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and finally, by differentiating,

$$\frac{\partial^2 i}{\partial r^2} + \frac{1}{r} \frac{\partial i}{\partial r} = \frac{4\pi\mu}{\rho} \frac{\partial i}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Writing

$$m^2 = \frac{4\pi\mu\omega}{\rho} = \frac{8\pi^2\mu f}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where f is the frequency of the alternating currents, we see that the value of i is given by (58).

Let us suppose that the current-density along the axis of the cylinder is given by $i = i_0 \cos \omega t$. At the axis r is zero; and since ker 0 is infinite both C and D must be zero, as otherwise the current-density would be infinite. Since also $\operatorname{ber} 0 = 1$ and $\operatorname{bei} 0 = 0$, we have $\mathrm{A} = i_0$ and $\mathrm{B} = 0$, and thus

$$i = i_0 \operatorname{ber} mr \cos \omega t - i_0 \operatorname{bei} mr \sin \omega t. \quad . \quad . \quad . \quad (65)$$

It is easy to verify by the help of the equations

$$\int r \operatorname{ber} mr \partial r = (r/m) \operatorname{bei}' mr \quad . \quad . \quad . \quad . \quad (66)$$

and

$$\int r \operatorname{bei} mr \partial r = -(r/m) \operatorname{ber}' mr \quad . \quad . \quad . \quad (67)$$

that this value of i, which is the same as that given by Kelvin, satisfies (62). It therefore gives the solution of the problem of finding the current-density at any point in the substance of the inner conductor.

From (65) we find that

$$\mathrm{I}_r = 2\pi \int_0^r ir\partial r$$

$$= (2\pi/m)\, r \operatorname{bei}' mr \,.\, i_0 \cos \omega t + (2\pi/m)\, r \operatorname{ber}' mr \,.\, i_0 \sin \omega t. \quad (68)$$

Putting r equal to a in this equation, we get for the total current,

$$\mathrm{I} = (2\pi/m)\, a \operatorname{bei}' ma \,.\, i_0 \cos \omega t + (2\pi/m)\, a \operatorname{ber}' ma \,.\, i_0 \sin \omega t. \quad (69)$$

By differentiating this equation with regard to t, we find that

$$\frac{\partial \mathrm{I}}{\partial t} = -(2\pi\omega/m)\, a \operatorname{bei}' ma \,.\, i_0 \sin \omega t$$

$$+ (2\pi\omega/m)\, a \operatorname{ber}' ma \,.\, i_0 \sin \omega t, \quad (70)$$

and hence, solving these equations for $i_0 \cos \omega t$ and $i_0 \sin \omega t$, we get

$$i_0 \cos \omega t = \frac{m \operatorname{bei}' ma}{2\pi a \mathrm{Y}(ma)} \mathrm{I} + \frac{m \operatorname{ber}' ma}{2\pi a\omega \mathrm{Y}(ma)} \frac{\partial \mathrm{I}}{\partial t}, \quad . \quad (71)$$

and

$$i_0 \sin \omega t = \frac{m \operatorname{ber}' ma}{2\pi a \mathrm{Y}(ma)} \mathrm{I} - \frac{m \operatorname{bei}' ma}{2\pi a\omega \mathrm{Y}(ma)} \frac{\partial \mathrm{I}}{\partial t}. \quad . \quad (72)$$

Let us now consider the currents in the outer conductor. Let i' be the current density in this cylinder at a distance r from the axis. Let e' be the potential-difference per unit length measured from the distributing station to the alternator, i' being considered positive when flowing in the same direction.

We have, therefore,

$$e' = (\rho/2\pi r\partial r)(2\pi r\partial r \,.\, i') - \partial\phi'/\partial t$$

$$= \rho i' - \partial\phi'/\partial t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

where

$$\phi' = \mu \int_r^c \frac{2(\mathrm{I} - \mathrm{I}'_x)}{x} \partial x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Thus $$r\frac{\partial\phi'}{\partial r} = -2\mu(\mathrm{I}-\mathrm{I}'_r)$$

$$= -4\pi\mu\int_r^c ri'\partial r. \quad . \quad . \quad , \quad . \quad . \quad . \quad . \quad (75)$$

Also, since we have supposed that e' does not vary with r, we get from (73) and (75)

$$\rho\frac{\partial i'}{\partial t} = -\frac{4\pi\mu}{r}\int_r^c r\frac{\partial i'}{\partial t}\partial r; \quad . \quad . \quad . \quad (76)$$

and thus $$\frac{\partial^2 i'}{\partial r^2}+\frac{1}{r}\frac{\partial i'}{\partial t} = \frac{m^2}{\omega}\frac{\partial i'}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

By (58), the solution of (77) may be written as follows:—

$$i' = (\mathrm{A}\ \mathrm{ber}\ mr+\mathrm{B}\ \mathrm{bei}\ mr+\mathrm{C}\ \mathrm{ker}\ mr+\mathrm{D}\ \mathrm{kei}\ mr)\, i_0 \cos\omega t$$
$$+(-\mathrm{A}\ \mathrm{bei}\ mr+\mathrm{B}\ \mathrm{ber}\ mr-\mathrm{C}\ \mathrm{kei}\ mr+\mathrm{D}\ \mathrm{ker}\ mr)\, i_0 \sin\omega t,$$
$$. \quad . \quad . \quad . \quad (78)$$

where A, B, C, and D are constants which have to be determined from the data of the problem.

Now noticing that

$$\int r\ \mathrm{ker}\ mr\,\partial r = (r/m)\ \mathrm{kei}'\ mr \quad . \quad . \quad . \quad (79)$$

and $$\int r\ \mathrm{kei}\ mr\,\partial r = -(r/m)\ \mathrm{ker}'\ mr, \quad . \quad . \quad (80)$$

we find, by substituting the value of i' given by (78) in (76), that

$$\mathrm{A}\ \mathrm{bei}'\ mc-\mathrm{B}\ \mathrm{ber}'\ mc+\mathrm{C}\ \mathrm{kei}'\ mc-\mathrm{D}\ \mathrm{ker}'\ mc = 0, \quad (81)$$

and

$$\mathrm{A}\ \mathrm{ber}'\ mc+\mathrm{B}\ \mathrm{bei}'\ mc+\mathrm{C}\ \mathrm{ker}'\ mc+\mathrm{D}\ \mathrm{kei}'\ mc = 0. \quad (82)$$

By equating also the integral value $\int_b^c 2\pi ri'\partial r$ of the current in the return conductor to the value of I given by (69), we get

$$\mathrm{A}\ \mathrm{bei}'\ mb-\mathrm{B}\ \mathrm{ber}'\ mb+\mathrm{C}\ \mathrm{kei}'\ mb-\mathrm{D}\ \mathrm{ker}'\ mb = -\frac{a}{b}\ \mathrm{bei}'\ ma, \quad (83)$$

and

$$\mathrm{A}\ \mathrm{ber}'\ mb+\mathrm{B}\ \mathrm{bei}'\ mb+\mathrm{C}\ \mathrm{ker}'\ mb+\mathrm{D}\ \mathrm{kei}'\ mb = -\frac{a}{b}\ \mathrm{ber}'\ ma. \quad (84)$$

The four equations (81)–(84) completely determine the four constants A, B, C, and D. Hence the current density at all points on the outer conductor is found.

From (73) and (74) we can see at once that

$$e' = \rho i'_b - \mu \frac{\partial}{\partial t} \int_b^c \frac{2(\mathrm{I} - \mathrm{I}'_x)}{x} \partial x, \quad . \quad . \quad . \quad (85)$$

where i'_b is the current density on the inner surface of the outer conductor. We also see from (59) and (61) that

$$e = \rho i_a + 2\mu' \log \frac{b}{a} \frac{\partial \mathrm{I}}{\partial t} + \mu \frac{\partial}{\partial t} \int_b^c \frac{2(\mathrm{I} - \mathrm{I}'_x)}{x} \partial x, \quad . \quad (86)$$

where i_a is the current density on the outer surface of the inner conductor. Thus, by addition,

$$e + e' = \rho i_a + \rho i'_b + 2\mu' \log \frac{b}{a} \frac{\partial \mathrm{I}}{\partial t}. \quad . \quad . \quad . \quad (87)$$

Hence writing for i_a and i'_b their values given by (65) and (68) respectively, and also writing for $i_0 \cos \omega t$ and $i_0 \sin \omega t$ their values in terms of I and $\partial \mathrm{I}/\partial t$ from (71) and (72), we get, after a little reduction, that

$$e + e' = \mathrm{RI} + \mathrm{L} \frac{\partial \mathrm{I}}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad (88)$$

where

$$\begin{aligned} \mathrm{R} = {} & (\rho m / 2\pi a \mathrm{Y}_a)(\mathrm{ber}\, ma\, \mathrm{bei}'\, ma - \mathrm{bei}\, ma\, \mathrm{ber}'\, ma) \\ & + \frac{\rho m\, \mathrm{bei}'\, ma}{2\pi a \mathrm{Y}_a} \{\mathrm{A}\, \mathrm{ber}\, mb + \mathrm{B}\, \mathrm{bei}\, mb + \mathrm{C}\, \mathrm{ker}\, mb + \mathrm{D}\, \mathrm{kei}\, mb\} \\ & - \frac{\rho m\, \mathrm{ber}'\, ma}{2\pi a \mathrm{Y}_a} \{\mathrm{A}\, \mathrm{bei}\, mb - \mathrm{B}\, \mathrm{ber}\, mb + \mathrm{C}\, \mathrm{kei}\, mb - \mathrm{D}\, \mathrm{ker}\, mb\}, \\ & \quad . \quad . \quad . \quad . \quad (89) \end{aligned}$$

and

$$\begin{aligned} \mathrm{L} = {} & 2\mu' \log \frac{b}{a} + \frac{2\mu}{ma \mathrm{Y}_a} (\mathrm{ber}\, ma\, \mathrm{ber}'\, ma + \mathrm{bei}\, ma\, \mathrm{bei}'\, ma) \\ & + \frac{2\mu\, \mathrm{ber}'\, ma}{ma \mathrm{Y}_a} \{\mathrm{A}\, \mathrm{ber}\, mb + \mathrm{B}\, \mathrm{bei}\, mb + \mathrm{C}\, \mathrm{ker}\, mb + \mathrm{D}\, \mathrm{kei}\, mb\} \\ & + \frac{2\mu\, \mathrm{bei}'\, ma}{ma \mathrm{Y}_a} \{\mathrm{A}\, \mathrm{bei}\, mb - \mathrm{B}\, \mathrm{ber}\, mb + \mathrm{C}\, \mathrm{kei}\, mb - \mathrm{D}\, \mathrm{ker}\, mb\}. \\ & \quad . \quad . \quad . \quad . \quad (90) \end{aligned}$$

In (89) and (90), Y_a stands for $\mathrm{ber}'^2\, ma + \mathrm{bei}'^2\, ma$, and A, B, C, and D can be found as follows from the equations (81)–(84), using the notation adopted in formulæ (54) and (55):—

$$\mathrm{AY}_c = -\mathrm{CS}_c + \mathrm{DT}_c \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

and

$$\mathrm{BY}_c = -\mathrm{CT}_c - \mathrm{DS}_c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Hence substituting the values of A and B, given by (91) and (94), in (83) and (84), we get

$$C(Y_c S_b - Y_b S_c) + D(Y_b T_c - Y_c T_b)$$

$$= -\frac{a}{b} Y_c (\mathrm{ber}' \, ma \, \mathrm{ber}' \, mb + \mathrm{bei}' \, ma \, \mathrm{bei}' \, mb), \quad (93)$$

and

$$-C(Y_b T_c - Y_c T_b) + D(Y_c S_b - Y_b S_c)$$

$$= -\frac{a}{b} Y_c (\mathrm{ber}' \, ma \, \mathrm{bei}' \, mb - \mathrm{bei}' \, ma \, \mathrm{ber}' \, mb). \quad (94)$$

From these equations C and D are easily found, and then A and B can be found from (91) and (92).

The above formulæ give the complete solution of the problem when the applied wave is sine-shaped and the capacity and leakage effects are neglected.

If the applied wave be not sine-shaped, but if it be a periodic function of the time, it may be expanded in a series of sines by Fourier's theorem; and hence we could write down the solution without difficulty, but it would be very cumbrous.

IV. Simplified Formulæ for particular cases.

1. *With direct currents.*

From the next solution, by putting m equal to zero, we find that the resistance R_d and the inductance L_d with direct currents are given by

$$R_d = \frac{\rho}{\pi a^2} + \frac{\rho}{\pi(c^2 - b^2)}, \quad . \quad . \quad . \quad . \quad (95)$$

and

$$L_d = 2\mu' \log \frac{b}{a} + \frac{\mu}{2}$$

$$+ \frac{2\mu c^4}{(c^2 - b^2)^2} \log \frac{c}{b} - \mu \frac{3c^2 - b^2}{2(c^2 - b^2)}. \quad (96)$$

This value for the inductance agrees with that found by Lord Rayleigh*.

* Phil. Mag. [5] vol. xxi. p. 381 (1886), or Russell's 'Alternating Currents,' vol. i. p. 53.

2. *With low frequency currents.*

Substituting the appropriate approximate formulæ (54) and (55) for S_c and T_c in the equations (91)–(94), and noticing that $Y_c=(m^2c^2/4)(1+m^4c^4/192)$ approx., we find that

$$C = C_1m^4 + C_2m^8, \quad \ldots \ldots \ldots \quad (97)$$

where

$$C_1 = \frac{a^2c^2(b^2-a^2)}{16(c^2-b^2)} - \frac{a^2b^2c^4}{4(c^2-b^2)^2}\log\frac{c}{b},$$

and

$$C_2 = \frac{c^2a^2}{2^2.4^2.6^2.8(c^2-b^2)}\left\{\begin{array}{l} b^2c^2(61b^2+19c^2-42a^2) \\ \quad +(b^2-a^2)(5a^2b^2-a^4-7b^4)\end{array}\right\}$$

$$-\frac{a^2b^2c^4\log(c/b)}{2^2\,4^2.12(c^2-b^2)^2}\left\{14c^2b^2-a^4-2b^4+3a^2b^2\right\}$$

$$-\frac{a^2b^4c^6(b^2-a^2)}{4^3(c^2-b^2)^3}\left\{\log\frac{c}{b}\right\}^2$$

$$+\frac{a^2b^6c^8}{4^2(c^2-b^2)^4}\left\{\log\frac{c}{b}\right\}^3.$$

$$-D = D_1m^2 + D_2m^6, \quad \ldots \ldots \ldots \quad (98)$$

where

$$D_1 = \frac{a^2c^2}{2(c^2-b^2)}$$

$$D_2 = \frac{a^2c^2}{2^2.4^2.6(c^2-b^2)}(7c^2b^2-a^4-2b^4+3a^2b^2)$$

$$+\frac{a^2b^2c^4(b^2-a^2)}{2.4^2.(c^2-b^2)^2}\log\frac{c}{b}$$

$$-\frac{a^2b^4c^6}{8(c^2-b^2)^3}\left(\log\frac{c}{b}\right)^2$$

$$A = \frac{a^2}{c^2-b^2} - \frac{\pi a^2c^2}{8(c^2-b^2)}m^2 + \frac{7a^2c^4m^4}{2^2.4^2.3(c^2-b^2)} + \frac{2}{c^2}D_2m^4$$

$$-C_1(\alpha+\tfrac{3}{4}-\log mc)m^4, \quad \ldots \ldots \quad (99)$$

and

$$\frac{B}{m^2}=\frac{2C_1}{c^2}+D_1(\alpha+\tfrac{3}{4}-\log mc)-\frac{\pi}{4}C_1m^2$$

$$+\left\{\frac{7c^2}{4^2.6}C_1+\frac{2}{c^2}C_2+D_2(\alpha+\tfrac{3}{4}-\log mc)+\frac{19c^4}{2^2.4^2.6^2.2}D_1\right\}m^4 \quad . \quad . \quad . \quad (100)$$

Substituting these values in (89) and (90), and using the approximate formulæ for the functions, we get, after lengthy algebraical operations,

$$R=\frac{\rho}{\pi a^2}\left(1+\frac{m^4a^4}{192}\right)$$

$$+\frac{\rho}{\pi(c^2-b^2)}\left\{1+m^4(\rho_1+\rho_2\xi+\rho_3\xi^2)\right\}, \quad . \quad . \quad (101)$$

where

$$\rho_1=\frac{(7c^2-b^2)(c^2-b^2)}{192},\ \rho_2=\frac{b^2c^4}{8(c^2-b^2)},$$

$$\rho_3=-\frac{b^2c^6}{4(c^2-b^2)^2},\ \text{and}\ \xi=\log\frac{c}{b}.$$

It is interesting to notice that the coefficient of m^2 vanishes and that the numerical constants α and $\log m$ have cancelled out. This formula is in exact agreement with that* given by Oliver Heaviside. It is not quite clear how he obtains the formula, but he states that the work was "very heavy."

In power transmission cables $a^2=c^2-b^2$. In this case the greater the value of b the smaller the value of $\rho_1+\rho_2\xi+\rho_3\xi^2$. In power transmission cables, when the frequency is not greater than 50, the increase in the effective resistance of the outer conductor due to the skin effect is negligibly small.

We find in a similar manner that

$$L=L_d-\frac{\mu}{2}\cdot\frac{a^4}{192}m^4$$

$$-\mu(\lambda_1+\lambda_2\xi+\lambda_3\xi^2+\lambda_4\xi^3)m^4, \quad . \quad . \quad (102)$$

* 'Electrical Papers,' vol. ii. p 192 (72).

where
$$\lambda_1=\frac{19c^6+103c^4b^2-41b^4c^2+3b^6}{2^2.4^2.6^2(c^2-b^2)}$$

$$\lambda_2=-\frac{14b^2c^4(2c^2-b^2)}{2^2.4^2.3(c^2-b^2)^2},$$

$$\lambda_3=-\frac{b^4c^6}{4(c^2-b^2)^3},$$

$$\lambda_4=\frac{b^4c^8}{2(c^2-b^2)^4}, \quad \text{and} \quad \xi=\log\frac{c}{b}.$$

Heaviside* gives the formula as if the coefficient of m^4 were zero. The term $-(\mu a^4/384)m^4$ can be deduced from the formula given in Gray and Matthews' 'Bessel's Functions,' p. 160, as the "self-inductance" of a cylindrical conductor.

Formula (102) shows that at low frequencies the inductance diminishes as the frequency increases, the effect being more pronounced the thicker the shell of the outer conductor.

3. *With high frequency currents.*

When ma is greater than 5 we can use the formulæ (50) to (53) for the ker functions and the corresponding formulæ for the ber functions. Substituting these values in (89) and (90) we get, after a little reduction,

$$R=\frac{\rho m}{2\pi a}\left\{\frac{1}{\sqrt{2}}+\frac{1}{2ma}+\frac{3}{8\sqrt{2}m^2a^2}\right\}$$
$$+\frac{\rho m}{2\pi b\sqrt{2}}\cdot\frac{\sinh m(c-b)\sqrt{2}+\sin m(c-b)\sqrt{2}}{\cosh m(c-b)\sqrt{2}-\cos m(c-b)\sqrt{2}}, \quad (103)$$

and

$$L=2\mu'\log\frac{b}{a}+\frac{2\mu}{ma}\left(\frac{1}{\sqrt{2}}-\frac{3}{8\sqrt{2}m^2a^2}-\frac{3}{8m^3a^3}\right)$$
$$+\frac{2\mu}{mb\sqrt{2}}\cdot\frac{\sinh m(c-b)\sqrt{2}-\sin m(c-b)\sqrt{2}}{\cosh m(c-b)\sqrt{2}+\cos m(c-b)\sqrt{2}}. \quad (104)$$

* 'Electrical Papers,' vol. ii. p. 192. [Dr. Heaviside has pointed out to me that he was only giving the first term correction to $R+L\omega\iota$, and thus it was unnecessary to give the coefficient of m^4 in (102).]

It is to be noticed that R becomes infinite when $b=c$, but L becomes

$$2\mu' \log (b/a) + (2\mu/ma)(1/\sqrt{2} - 3/8\sqrt{2}m^2a^2 - 3/8m^3a^3).$$

The first term in (103) gives the resistance of the inner conductor and the second that of the outer conductor. In (104) the first term gives the linkages of the magnetic flux in the dielectric with the current in the inner conductor. The second term gives the linkages of the magnetic flux in the substance of the inner conductor, and the third the linkages of the flux in the outer conductor.

4. *With very high frequency currents.*

When the frequency is very high, provided that c be not nearly equal to b, we may write

$$\begin{aligned} R &= \frac{\rho m}{2\pi a\sqrt{2}} + \frac{\rho m}{2\pi b\sqrt{2}} \\ &= \sqrt{\mu f \rho}\left(\frac{1}{a} + \frac{1}{b}\right). \quad \ldots \quad (105) \end{aligned}$$

Similarly when ma is very great we may write

$$L = 2\mu' \log \frac{b}{a} + \frac{\mu}{2\pi}\sqrt{\frac{\rho}{\mu f}}\left(\frac{1}{a} + \frac{1}{b}\right). \quad . \quad (106)$$

We see, therefore, that as f increases R continually increases, but L approaches the value $2\mu' \log (b/a)$ asymptotically. Whatever the frequency, we see that the value of L lies between the value given by (96) and $2\mu' \log (b/a)$. It is easy to see that it has the latter value when the currents are confined to an infinitely thin skin on the outer and inner surfaces of the inner and outer conductors respectively.

Formulæ (105) and (106) agree with those given by Sir Joseph Thomson (see 'Recent Researches,' p. 295). They are also given in Heaviside's 'Electrical Papers,' vol. ii. p. 193.

V. The Density of the Current in the Inner and Outer Conductors.

1. *With low frequency currents.*

For the inner conductor we find from equation (65), that

$$i = i_0 \{X(mr)\}^{1/2} \cos(\omega t + \epsilon), \quad . \quad . \quad . \quad (107)$$

where $\tan \epsilon = \text{bei}\, mr / \text{ber}\, mr.$

It is known * that $X(mr)$ always increases as mr increases. Hence the amplitude of the current density is always greatest at the surface of the inner conductor and least along the axis. When the sixth and higher powers of mr can be neglected (107) becomes

$$i = i_0(1 + m^4r^4/64) \cos(\omega t + \epsilon), \quad . \quad . \quad . \quad (108)$$

where $\tan \epsilon = m^2r^2/4.$

If $mr = 1$, (107) gives

$$i = 1{\cdot}015\, i_0 \cos(\omega t + 14^\circ\ 15'),$$

and (108) gives $i = 1{\cdot}016\, i_0 \cos(\omega t + 14^\circ\ 2').$

Hence when mr is not greater than 1, (108) may be used. Even when $mr = 2$ the inaccuracy in the value of the amplitude of the current density given by (108) is less than 2 per cent. In this case the amplitude of the current-density at the surface of the conductor is about 23 per cent. greater than at its axis, and the current along the axis lags by about 52° behind the surface current.

Similarly when mc is small, we find by (78) that

$$\frac{i'}{i_0} = \frac{a^2}{c^2 - b^2} \left\{ 1 - \frac{m^2c^2}{2} \log mr \right\} \cos \omega t. \quad . \quad (109)$$

Hence the amplitude of the current-density in the outer conductor diminishes as r increases.

3. *With high frequency currents.*

In this case, by (20) and (21), we get for the inner conductor

$$\frac{i}{i_0} = \frac{\epsilon^{mr/\sqrt{2}}}{\sqrt{2\pi mr}} \cos\left(\omega t + \frac{mr}{\sqrt{2}} - \frac{\pi}{8}\right). \quad . \quad . \quad (110)$$

* Russell's 'Alternating Currents,' vol. i. p. 373.

Similarly in the outer conductor we find that

$$i_0 = \frac{\epsilon^{ma/\sqrt{2}}}{\sqrt{2\pi mr}}\left(\frac{a}{b}\right)^{1/2}\left[\frac{\cosh m(c-r)\sqrt{2}+\cos m(c-r)\sqrt{2}}{\cosh m(c-b)\sqrt{2}-\cos m(c-b)\sqrt{2}}\right]^{1/2}\cos(\omega t+\theta), \quad . \quad . \quad . \quad (111)$$

where

$$\tan\theta = \frac{\epsilon^{-m(c-r)/\sqrt{2}}\sin\epsilon_1+\epsilon^{m(c-r)/\sqrt{2}}\sin\epsilon_2}{\epsilon^{-m(c-r)/\sqrt{2}}\cos\epsilon_1+\epsilon^{m(c-r)/\sqrt{2}}\cos\epsilon_2},$$

and

$$\epsilon_1=\gamma-\frac{mr}{\sqrt{2}}+\frac{\pi}{8};\quad \epsilon_2=\delta+\frac{mr}{\sqrt{2}}+\frac{\pi}{8};$$

$$\tan\gamma=\frac{B}{A};\quad \tan\delta=\frac{D}{C},\quad \text{and } \gamma-\delta=mc\sqrt{2}.$$

Since $\{\cosh m(c-r)\sqrt{2}+\cos m(c-r)\sqrt{2}\}/r$

continually diminishes as r increases, we see that the amplitude of i always diminishes as r increases.

VI. Concentric Main with Hollow Inner Conductor.

Let us suppose that a_1 is the inner radius of the inner conductor. The solution (58) for the current-density i still applies. Assuming that $i=i_0\cos\omega t$, when r is a_1, we get two equations connecting the four constants A, B, C, and D. The equation corresponding to (62) is now

$$\rho\frac{\partial i}{\partial r}=\frac{4\pi\mu}{r}\int_{a_1}^{r} r\frac{\partial i}{\partial t}\partial r. \quad . \quad . \quad (112)$$

Substituting the value of i in this equation and equating the coefficients of $\cos\omega t$ and $\sin\omega t$ in the resulting equation to zero, we get two other equations connecting the four constants, and hence they can be found.

Adopting the method suggested by O. Heaviside*, however, the values of the effective resistance of each tube may be written down at once from the formulæ given above. From formula (101), for instance, the resistance R_0 of the outer

* 'Electrical Papers,' vol. ii. p. 192.

conductor is given by

$$R_0=\frac{\rho}{\pi(c^2-b^2)}\left[1+m^4\left\{\frac{(7c^2-b^2)(c^2-b^2)}{192}+\frac{b^2c^4}{8(c^2-b^2)}\log\frac{c}{b}\right.\right.$$
$$\left.\left.-\frac{b^2c^6}{4(c^2-b^2)^2}\left(\log\frac{c}{b}\right)^2\right\}\right]. \quad . \quad . \quad (113)$$

To deduce the resistance R_i of the inner tube we write a for b, and a_1 for c in the coefficient of m^4, and noticing that the area of the cross section is now $\pi(a^2-a_1^2)$, we get

$$R_i=\frac{\rho}{\pi(a^2-a_1^2)}\left[1+m^4\left\{\frac{(7a_1^2-a^2)(a_1^2-a^2)}{192}+\frac{a^2a_1^4}{8(a^2-a_1^2)}\log\frac{a}{a_1}\right.\right.$$
$$\left.\left.-\frac{a^2a_1^6}{4(a^2-a_1^2)^2}\left(\log\frac{a}{a_1}\right)^2\right\}\right]. \quad . \quad (114)$$

For a given cross sectional area $\pi(a^2-a_1^2)$ we see from (114) that the coefficient of m^4 diminishes as a_1 increases. Hence, at low frequencies, making the inner conductor hollow diminishes the skin effect. Similarly for a given cross sectional area, the larger the outer tube the smaller the effective resistance of the tube.

From (103) we see that at high frequencies the resistance R_0 of the outer tube is given by

$$R_0=\frac{\rho m}{2\pi b\sqrt{2}}\cdot\frac{\sinh m(c-b)\sqrt{2}+\sin m(c-b)\sqrt{2}}{\cosh m(c-b)\sqrt{2}-\cos m(c-b)\sqrt{2}}.$$

Hence the resistance R_i of the inner tube is given by

$$R_i=\frac{\rho m}{2\pi a\sqrt{2}}\cdot\frac{\sinh m(a-a_1)\sqrt{2}+\sin m(a-a_1)\sqrt{2}}{\cosh m(a-a_1)\sqrt{2}-\cos m(a-a_1)\sqrt{2}}. \quad (115)$$

The difference, therefore, between the effective resistance of a thin and a thick inner tube having equal outer radii is very small at high frequencies.

VII. The Impedance of a Concentric Main.

If the length of the main be very long compared with its diameter and the insulation resistance of the dielectric be very high, the assumption that the current flow is linear is permissible. In this case, if e be the potential-difference

between the mains at a point at a distance x from the alternator, we have*

$$-\frac{\partial e}{\partial x}=\mathrm{R}i+\mathrm{L}\frac{\partial i}{\partial t},$$

and

$$-\frac{\partial i}{\partial x}=\frac{e}{\mathrm{S}}+\mathrm{K}\frac{\partial e}{\partial t},$$

where K and S are the capacity and the insulation resistance between the mains per unit length. In these equations R and L have the values found earlier in the paper. Hence we have to substitute these values in the well-known expressions for the impedance of the main.

VIII. Numerical example.

In connexion with the transmission of electric power at low frequencies, the question of the magnitude of the skin effect in three-core cables has been recently discussed by engineers. As the question is one of considerable difficulty owing to the very complex nature of the magnetic field inside a three-core cable (see Russell, 'Alternating Currents,' vol. i. p. 321), it is important to know what are the corresponding losses in a concentric main.

Let us suppose that the inner core is solid and that mc is not greater than 2, so that we may use formula (101). Let the radius of the inner main be one centimetre. For a very high pressure cable a suitable value† of b would be 2·4 cms. Hence, since the section of the outer conductor is made equal to that of the inner, c is 2·6 cms. Substituting these numbers in the formula, we find that

$$\frac{\mathrm{R}}{l}=\frac{\rho}{\pi}\left(1+\frac{m^4}{192}\right)+\frac{\rho}{\pi}\left(1+\frac{0{\cdot}0072\,m^4}{192}\right),$$

where l is the length of the main in centimetres.

With the frequencies used in practice m is not very different from 1. We see that the skin effect increases the resistance of the inner conductor by about the half of one per cent., and the increase in the effective resistance of the

* Russell, 'Alternating Currents,' vol. ii. p. 458 *et seq.*

† Russell, 'Electric Cables and Networks,' p. 203.

outer conductor is less than the hundredth part of this. For a low voltage cable we might have $c=5/3$ and $b=4/3$. In this case

$$\frac{R}{l}=\frac{\rho}{\pi}\left(1+\frac{m^4}{192}\right)+\frac{\rho}{\pi}\left(1+\frac{0{\cdot}059\,m^4}{192}\right),$$

and hence the increase in the loss of the outer conductor is only about the twentieth part of the corresponding quantity for the inner. If the return conductor were a tight-fitting tube so that $b=1$ and $c=\sqrt{2}$, the formula becomes

$$\frac{R}{l}=\frac{\rho}{\pi}\left(1+\frac{m^4}{192}\right)+\frac{\rho}{\pi}\left(1+\frac{0{\cdot}148\,m^4}{192}\right).$$

Even in this case the increase in the loss of the outer is less than the fifth part of the increase in the loss of the inner, although the losses for very high frequencies would be practically the same in each conductor.

If the main consisted of a hollow inner conductor whose radii were 1 and $\sqrt{2}$ respectively and an outer conductor of radii 2·6 and 2·4, then we would have

$$\frac{R}{l}=\frac{\rho}{\pi}\left(1+\frac{0{\cdot}105\,m^4}{192}\right)+\frac{\rho}{\pi}\left(1+\frac{0{\cdot}0072\,m^4}{192}\right).$$

Hence, making the inner conductor hollow has appreciably reduced the losses due to skin resistance. It will also be noticed that the losses due to this cause in the $(1, \sqrt{2})$ cylindrical conductor are about 30 per cent. less when the return current is outside it than when it is inside it.

It is easy to see that if the current returns by a fine conductor in a minute hole along the axis of a cylindrical conductor, the increase in the losses for low-frequency currents will be nearly seven times greater than if it returned by a concentric tube outside the conductor.

IX. Values of m for Copper Conductors.

The volume resistivity of high conductivity annealed copper at 60° F. (15°·6 C.) is 1696·5. Hence

$$m^2=\frac{4\pi\mu\omega}{\rho}=\frac{8\pi^2\mu}{1696{\cdot}5}f.$$

Assuming that $\mu = 1$, we get the following table :—

f.	*m*.	*f*.	*m*.
5	0·4824	100	2·1573
10	0·6822	200	3·0509
15	0·8355	300	3·7366
20	0·9648	400	4·3147
25	1·0787	500	4·8240
30	1·1816	1000	6·8221
35	1·2763	5000	15·255
40	1·3644	10000	21·573
45	1·4472	100000	68·221
50	1·5255	1000000	215·734

Discussion.

Mr W. Duddell congratulated the Author, and remarked that the results obtained in the paper would be most useful. It was, he said, customary in dealing with high-frequency currents to use several parallel insulated wires instead of a single wire in order to increase the surface and obtain greater conductivity. He asked the Author to what extent it was advisable to do this. He also asked how far it was valuable to silver-plate copper wires for high-frequency currents.

Mr A. Campbell observed with regard to Dr. Russell's warning that inductance standards might be greatly dependent on frequency, that well designed single-layer wavemeter coils of highly stranded wire (*e. g.* 7/36^3) did not show according to his experiments a variation of more than 1 or 2 in 1000 in their self-inductance for frequencies from 0 up to 1,000,000 $\sim$ per second, the actual inductances being from 10 to 200 microhenries.

Dr W. H. Eccles remarked that in high-frequency work the resistance of a conductor depended on the nature of the surface. If a copper wire tarnished its resistance altered. Uniform results could be obtained by lacquering the wires. He congratulated the Author upon the mathematical results obtained.

Mr B. S. Cohen said that a knowledge of the variations with frequency of effective resistance and inductance of copper conductors was of considerable importance in connexion with telephonic transmission, and it was to be hoped that Dr Russell would extend his valuable investigations to conductors lying side by side. The heaviest gauge of conductors met with in general telephonic practice is 2·85 mm. in diameter in the case of cables, and these conductors are separated by about 2·5 mm. of combined paper and air dielectric and twisted together with a lay of about 15 cms.

In the case of overhead open wires, the largest conductor in general use is 4 mm. in diameter, although in a number of cases a 5·69 mm. conductor has been used. The mean axial distance apart of these conductors is about 35 cms. as they are rotated in sets of 4 in a foot square.

Applying Lord Rayleigh's formula for effective resistance in these cases we get, for a frequency of 1500 ~, an increase of 1 per cent. in effective resistance over steady current resistance in the case of the 2·85 mm. cable wire. A frequency of 1500 ~ can quite safely be taken as the maximum frequency which is of any importance in speech articulation. The increase with the 5·69 mm. open wire at this same frequency is 12 per cent., and with the 4 mm. wire it is 3·55 per cent. In a table given in a well-known Pocket Book, a 5·69 mm. conductor is stated to increase 15 per cent. in effective resistance at 1500 ~, and he noticed that this figure had been adopted by one writer on telephonic matters. Mr G. M. B. Shepherd some time ago worked out the decrease in inductance with frequency for wires lying side by side by both Heaviside's and Rayleigh's formulæ. The following percentage variations between 0 and 1500 ~ were found for 5·69 mm. wire:—L ·37 per cent. decrease; r ·22 per cent. increase. This is very small, and is due to the fact that most of the inductance is due to the field between wire and wire. On the other hand, increase of resistance in this wire at 1500 ~ means an increase in attenuation of 12 per cent.

In the case of 5·69 mm. conductors 33 cms. apart, the inductance as calculated by Maxwell's formula amounts to 3·2 millihenries, which would reduce the attenuation constant to one-fifth of its value at 1500 ~. It is very desirable therefore to know the order of decrease of inductance with frequency.

Lastly, I should like to ask Dr Russell whether he thinks that the presence of the adjacent conductors in a telephone cable or open wire route is likely to modify the skin effect to any decided extent.

Mr Paterson expressed his interest in the paper, and asked if the formula given for the effective resistance of the core of a concentric main would be of the same form if the outer layer were removed. He also asked if the skin effect in the outer shell would be altered if the shell were cut parallel to its length and laid flat.

Prof. C. H. Lees pointed out that a change of sign in one of the early equations would lead to the use of the Bessel J function instead of the I function.

Dr Drysdale congratulated the Author, and wished to make a suggestion from the point of view of facilitating the use of these and other calculations. Electrical measurements frequently necessitated the employment of higher functions, and this involved either troublesome calculations or the compilation of bulky tables. He had therefore been led to try whether graphical methods could not be employed. The plotting of a function $f(x)$ graphically could only give values to a low degree of accuracy, but if some simple empirical formula $y=\phi(x)$ were found which approximated to the required function, it was possible to express the true value $f(x)$ in the form $r\phi(x)$ or $\phi(x)+a$,

where r or a might be termed a correcting factor or term, which only varied between comparatively narrow limits. By employing a curve in which $\log r$ or a was plotted against x, it was frequently possible to obtain values of a function from a single curve correct to 1 in 10,000 or closer, and which greatly assisted interpolation. He had found this of great service in dealing with elliptic integrals, and solid angles, and thought that it might perhaps also be applied to the ber and bei functions.

The AUTHOR, in reply to Mr Duddell, stated that in his opinion stranding and insulating the strands of the wires used in wireless telegraphy would not diminish the skin effect, if the strands were parallel to the axis of the wire. It would probably increase it. Silver-plating the wires would be beneficial. The distribution of the high-frequency current on the surface of the wires can sometimes be found by remembering that the distribution is such that no magnetic induction is produced in the metal. The corresponding electrostatic problems are consequently of great help. Mr Campbell's experimental results were interesting as they show that it is possible to make coils of low time constant so that frequencies even as high as a million have little effect on their inductance. These coils, however, are of little use for many experimental purposes. With ordinary laboratory coils a frequency of 10,000 will alter their inductance 2 or 3 per cent. If we have two parallel cylindrical wires practically touching, the inductance will alter from 3·7726 cms. per unit length to practically zero as the frequency is increased.

Dr Eccles rightly lays stress on the importance of keeping the surface of the wires used in wireless telegraphy in good condition and careful lacquering would be very beneficial. Mr Cohen's data about the telephone cables used in long distance transmission are most interesting. A piece of metal adjacent to the wire would in many cases modify the skin effect, but it would be difficult to deduce any general rule as the main current is affected in different ways by the induced eddy-currents in the neighbouring metal according to the relative positions and magnitudes of the metal and wire. Mr Paterson's question was rather hard to answer. In the hollow tube the current-density was a maximum at the outer surface. When split open and laid out flat, the current-density would be a maximum at the edges. In both cases, the uneven distribution of the current leads to an increase in the apparent resistance. But which would be the greater he could only determine by calculation. In reply to Professor Lees, he stated that he had first given the Kelvin functions in terms of Bessel's J function, but he had changed it to the I function as this made a + instead of a − in the fundamental definition. Dr Drysdale's methods of calculation would be convenient in the case of the elliptic functions, but it would not be so easy to apply them to periodic functions of continuously increasing amplitude.

XL. *A Laboratory Machine for Applying Bending and Twisting Moments simultaneously.* *By* Professor E. G. Coker, *M.A., D.Sc., of the City and Guilds of London Technical College, Finsbury* *.

[Plates XXV. & XXVI.]

Apparatus for applying simple tension or compression stresses, and also for applying bending or twisting moments to materials, are in common use in laboratories, but such machines are, as a rule, not very well adapted for experiments on combined stresses, although the frequency of such cases in practice make it very desirable that experiments on the effects of combinations of stresses should be carried out by engineering students. This is particularly the case with shafts which are generally subjected to stresses due to combined bending and twisting moments. The present paper describes a machine built by students of the City and Guilds Technical College, Finsbury, in which uniform bending and twisting moments can be applied simultaneously over the whole length of the specimen, and in any desired proportion to each other.

The machine is similar to one designed by the author for the testing laboratory at McGill University, Montreal, but with some modifications suggested by experience with the earlier machine. The principle on which the design is based is illustrated by fig. 1 (Pl. XXV.), in which a rod R is suspended at intermediate points A, B, by wires C, D, depending from a fixed support E. The equal overhanging ends of the rod are loaded by weights W, so that the applied couple between the points of support is uniform and of amount Wa, where a is the length of the lever-arm. The rod is also twisted by weights W_1 attached to equal arms of length b, so that there is a uniform twisting moment of amount W_1 between the points of suspension. The two systems of loading are independent and their ratio can be adjusted to any value desired.

In carrying out this arrangement in practice it is con-

* Read February 26, 1909.

venient to arrange that one of the levers for applying the twisting moment shall always remain in a horizontal position, and that the other shall be capable of turning through an arc to bring the first lever back to zero after each application of the load. The most convenient way of carrying this out is to replace the adjustable lever by a worm and worm-wheel gear secured in a casing and turned by a hand-wheel. To allow freedom for bending the worm-wheel casing must be pivoted to rotate around a line intersecting the axis of the specimen and perpendicular thereto, and this method of pivoting must also be adopted at the horizontal lever. This arrangement only differs from that of the perfectly freely suspended arrangement shown by fig. 1 (Pl. XXV.) in fixing one point of the rod, and this has the indirect advantage of stilling vibration which is troublesome in the freely suspended bar.

The arrangement described above is carried into effect in the manner indicated by figs. 2, 3, 4, and 5, showing the apparatus in side elevation, end elevations, and plan respectively.

The various parts are supported in a built up frame consisting of two planished steel shafts, A secured in cast-iron cross frames, B mounted on four standards, one of which latter is adjustable in height to secure steadiness on an uneven floor. Upon the steel shafts are two castings C, D, each of which has a cylindrical bearing E encircling one of the shafts and resting with a flat face F in line contact with the other shaft, and secured in position by a cross-bar G threaded on studs. This connexion is perfectly rigid, since it removes all degrees of freedom and it is readily released by simply turning back one of the cross-bar nuts, leaving the casting free to slide into a new position. It also has the advantage that no accurate fitting is required for the supporting frame. The casting C carrying the worm-wheel gear W has trunnion bearings H at right angles to and intersecting the axis of the specimen. The bearings are fitted with friction rollers, and when the machine is used simply for torsion the worm-wheel is kept in a vertical position by an arm I keyed to the bearing H and locked in position by a thumb-screw. A weight J attached by an arm to the second bearing balances the pivoted casing in all positions.

The weigh levers are supported from a vertical standard K of the frame D by a wire L, terminating in a thin plate M with a keyhole slot encircling the spindle N. Formerly a roller bearing was used for this spindle, but this is an unnessary refinement as the friction is extremely small, and can be easily taken into account. The casting supported in this way has three levers, P, Q, and R, the first two of which are for the application of twisting moments S, and the third R, in the line of the specimen, is for applying a bending moment.

All the loading levers are provided with knife-edges S, fig. 6, of circular form, made by turning an ordinary Whitworth nut down to form a disk with a V-shaped edge. These disks carry rings T with wide angled V-shaped recesses on the inner sides, and light rods V screwed into these rings carry the weights. This arrangement of knife edge is very easy to adjust accurately, and when bending and twisting stresses are applied simultaneously the rolling line contact adjusts itself to the bending and twisting of the specimen. The bending of the specimen causes a change in the effective arm of the bending levers, which is generally negligible, but a correction may be necessary with a very long specimen. For if a is the length of the lever-arm and b is the radius of the circular knife-edge, an angular deviation of amount θ will cause a change of $a-(a\cos\theta+b\sin\theta)$ in the lever-arm, and this is zero when $\theta=0$ and also when $a=a\cos\theta+b\sin\theta$.

In the machine described a is 10 inches and b is 0·5 inches, and the angles $\theta=0$ and $\theta=5^\circ{\cdot}75$ both correspond to an effective length of 10 inches. The maximum correction between these values is easily shown to be at an angle θ given by the equation $b\cos\theta=a\sin\theta$, in the present case $2^\circ{\cdot}9$ approximately, for which value the correction is 0·12 per cent. For values of θ greater than 6° the correction increases more rapidly, and its amount may be obtained from the diagram, fig. 7, which shows the percentage error for all angles up to 10°. In the majority of tests the angular change at the ends rarely exceeds 5°, and the correction is therefore so very small as to be practically negligible.

The worm-wheel W and the casting Y for the weigh-

levers are bored out to receive the ends of the specimen, and are provided with fixed keys which slide in corresponding key-ways cut in the specimen. When tubes are subjected to stress they are provided with solid ends secured by transverse pins, thereby avoiding brazed joints since these latter are troublesome owing to the state of the metal being altered by the brazing. The end of the specimen projecting through the worm-wheel is fitted with a lever X for applying bending moment, and both levers for bending may be loaded independently or by a cross-bar suspended from stirrups as shown in fig. 2.

A photograph of the machine is shown by fig. 8 (Pl. XXVI.) with a specimen inserted which has failed under the combined effect of bending moment and twisting moments.

Measurement of the Strains.

The worm-wheel is graduated in degrees and a vernier circle enables 0·1 of a degree to be read with ease, while in order to start with a zero reading this vernier is carried on a ring sliding in a groove in the casing so that it can be adjusted to any angular position. This arrangement measures the twist on the whole specimen, and includes any motion due to back-lash in the keys and keyways during a test, it is therefore only suitable for measurements with long wires having substantial ends, and for plastic strains in which the end effects are negligible in comparison.

For observations within the elastic limit the author prefers to use an instrument* which is secured to the specimen and is self-contained.

This instrument was originally designed to measure the angle of twist within the elastic limit, and with some recent alterations it can be adjusted in a few seconds for measuring the angular change due to bending. The calibration of the readings is effected on the specimen and serves for both bending and twisting. Fig. 9 (Pl. XXV.) shows the apparatus in part longitudinal section.

It consists of a graduated circle A mounted on the specimen

* "On Instruments for Measuring small Torsional Strains," Phil. Mag. December 1899.

B by three screws C in the chuck-plate D. A sleeve E provided with three screws grips the specimen at a fixed distance away from the first set.

The spacing of these two main pieces on the specimen is effected by a clamp, not shown in the figure, which grips the double cones F, G, and maintains them at the correct distance apart, until the set screws are adjusted.

The clamp is afterwards removed, leaving the plane of the graduated circle perpendicular to the axis of the specimen and the sleeve correctly set and ready to receive the reading-microscope H.

The vernier plate carries a sliding tube I, on which a wire J is mounted, and the movement of this latter due to bending or twist is measured by a scale in the eye-piece K, the divisions of which are calibrated by reference to the graduated circle. It is found convenient to have the microscope-tube pivoted about an axis perpendicular to its central line at L, so that any slight difference due to imperfect centering can be adjusted by the screw M to make the calibration value agree for a series of specimens.

The observation wire may be set at any convenient position for calibration, but for observations of the angle of twist when the specimen is also subjected to a uniform bending moment the wire should be in the central plane perpendicular to the specimen. For if the bending is in the plane containing the axis of the specimen and the observation wire, it has the effect of causing new parts of the wire to come into view on the scale, but no error is caused thereby. If the specimen is bent in a plane at right angles to the former, then the change in the reading is $(\theta-\phi)l$, where θ and ϕ are the alterations of angle at the ends and $2l$ is the length of the specimen under observation. Since the bending is uniform $\theta=\phi$ and no correction is necessary. Bending in any other plane can be resolved into components in the vertical and horizontal planes, and therefore falls under the preceding cases. In order to effect the adjustment required, both the wire and the microscope slide in adjustable tubes provided with graduated scales, and the movement to bring the wire into focus is divided between them.

To check the setting of the wire in the central position it is convenient to apply a uniform bending moment, and then to observe if any change takes place in the reading. The position for no change in the reading can be found in a few seconds.

In experiments where the bending moment is constant and the twisting moment is varied, no adjustment is practically required during the elastic life of the specimen; and even when the bending moment is variable the adjustment is practically negligible, as the length of the specimen under test is only a few inches.

The instrument is used for observations of the angular change due to bending by adjusting the wire in the horizontal plane passing through the axis of the specimen, and at a fixed distance away from the central plane, as shown in fig. 7. Thus if the wire is at a distance x from the central plane, and the specimen is subjected to a uniform bending moment, the reading will be $(l+x)\theta-(l-x)\theta=2x\theta$, and this is a measure of the angular change θ between the ends, since any correction involves higher powers of θ which are negligible for elastic strains.

The instrument may therefore be used for measuring strains due to bending or twisting, and the single calibration required for both sets of readings is effected when the instrument is in position on the specimen.

With the usual notation the value of Young's modulus can be calculated from the equation $EI_1 dy^2/dx^2=M$, from which we obtain $EI_1\theta_1=Ml$ for determining E. Similarly the rigidity modulus for specimens of circular sections may be calculated from the equation $C=Tl/I_2\theta_2$. The ratio of C to E can also be readily obtained from the value $a\theta_1/\theta_2$, where a is an instrument constant: values of Poisson's ratio may be determined in this way without calculating C and E separately. As an example, we may take the case of a test on a steel tube 6 inches long, having an external diameter of 0·748 inch and a wall-thickness of 0·024 inch.

The calibration value of the eyepiece scale where referred

to the graduated circle was found to be 10·35 divisions for one minute of arc. The following readings were obtained:—

Twisting Moment. Inch-Pounds.	Reading	Δ.
0	0	
100	237	−237
200	474	−237
300	712	−238
400	920	−238
500	1188	−238

and the value of C—12,400,000 in inches and pounds was determined from these observations.

The observation wire was then rotated into a horizontal position and set at a distance of one inch from the central plane. Bending moments were applied and readings were observed as follows :—

Bending Moment. Inch-Pounds.	Reading	Δ.
0	0	
100	59	−59
200	118	−59
300	178	−60
400	239	−61
500	297	−58

from which the value of E was calculated. The angular change θ_2 of one end of the specimen was determined from the readings R by the formula $R=2x\theta_2$, where $x=1$ inch, and account was taken of the fact that the effective radius of arc was changed from r_1 to r_2 in the new adjustment of the instrustment so that the angle θ_2 must be multiplied by r_1/r_2. In this instance r_1/r_2 was 1·61 and $E/C=r_2/r_1 \,.\, \theta_1/\theta_2=2{\cdot}485$ or $E=30{,}800{,}000$ and Poisson's ratio$=\cdot243$.

A tension-test on a similar piece of tube gave $E = 30,700,000$.

In addition to its applications for bending and twisting the apparatus may be used for testing a variety of cases of combined stress if a pump is added to give a fluid pressure in the interior of tubes.

The accompanying diagram (Pl. XXV. fig. 10) shows the results of tests to failure of bicycle-tubing when subjected to (I.) bending, (II.) twisting, and (III.) twisting combined with a uniform bending moment.

Discussion.

Prof. F. T. Trouton said he was interested in the experiments from the point of view of the viscous flow which occurred towards the end of the test. He asked if the flow would be different if the end of the bar was unloaded.

The Chairman expressed his interest in the paper, and asked if the experiments favoured any particular theory of fracture.

Prof. Coker said he had not yet made many experiments with the machine, but those made seemed to show that failure was due to shear stress. He hoped to carry out further experiments on both ductile and brittle materials.

XLI. *On the Self-Demagnetizing Factor of Bar Magnets.* *By* Silvanus P. Thompson, *D.Sc., F.R.S., and* E. W. Moss *.

[Plate XXVII.]

This paper consists of three parts :—(i.) A discussion of the significance and definition of the self-demagnetizing factor of magnets in general, and of bar-magnets in particular; (ii.) a redetermination of the values of the self-demagnetizing factor for bar-magnets of circular section; (iii.) determination of the values of the self-demagnetizing factor for bar-magnets of rectangular cross-sections of various proportions.

Part I.—Preliminary. On the Significance and Definition of the Self-Demagnetizing Factor.

Between any two magnet-poles, whether they are regarded as points, or as regions over which there is a surface-distribution of magnetism, there are magnetic forces. In

* Read February 26, 1909.

the space between any two point-poles the intensity of the magnetic field that is due to these poles, at any point in the line joining them, is expressed by the equation :

$$\mathscr{H}_x = \frac{m_1}{(a+x)^2} - \frac{m_2}{(a-x)^2} ;$$

where the respective strengths of poles are m_1 and $-m_2$; a the half of the distance between them, and x the distance of the point in question from the mid-point between them. The value of this expression in no way depends on the material in the space between the poles, whether non-magnetic or magnetic, or actually magnetized in any manner.

If m_1 and $-m_2$ are numerically equal, the expression becomes :

$$\mathscr{H}_x = 2m \frac{a^2 + x^2}{(a^2 - x^2)^2}.$$

At the mid-point, under the same condition, the intensity has the minimum value of

$$\mathscr{H}_{\text{min.}} = 2m \div a^2.$$

If the space between the two point-poles be regarded as occupied by a thin, cylindrical, uniformly-magnetized steel magnet the ends of which constitute the point-poles in question, then these equations will be the expresions for a self-produced magnetic field acting in a direction which opposes the actual magnetism of the magnet, and tending to demagnetize it. Each portion of the filiform magnet will be acted upon by a demagnetizing field, strongest towards the poles, weakest at the middle. The supposed uniform magnetization of the magnet will of course be unstable. If it were produced, even for a moment, there would at once be a retrocession of a portion of the magnetization from the ends, with a new distribution of the polarity. On the supposition that the middle part of the rod retains still its full flux, the retrocession of the pole would shorten the effective length of the magnet, diminishing the magnetic moment, but increasing any self-demagnetizing internal action. This tendency to produce a retrocession of the pole may operate to different degrees according to whether the bar consist of soft iron, or hard tungsten steel. In either case the retreat of the pole can be only incomplete : because if we suppose the pole to have

actually retreated by any given amount—for example 1 centimetre—the end piece of that length will now be subjected to the magnetizing action of the rest of the bar, and will be remagnetized up to a certain point, namely, such that the reaction of the magnetism of this piece is equal to the magnetizing action of the whole of the rest of the bar, less the demagnetizing reaction of the bar as a whole. The inevitable result is a distributed pole. It cannot remain concentrated at one point, on the end; it must redistribute itself along the bar with a distribution determined by the conditions of equilibrium at every point.

Also the middle piece of the bar will not be exempt from influence, it, too, must diminish its inherent magnetism, because even in weak fields the magnetism of the hardest steel is subject to cyclical changes; and because any retrocession of the poles is, *pro tanto*, productive of an increase in the self-demagnetizing force at the middle. Only in cases where this self-demagnetizing force at the middle is less than that which suffices to produce an irreversible change in the magnetism of the steel, that is only in cases where the bar is very long in proportion to its cross-section, can the action at the middle be regarded as negligible.

It is clear then, in general, that for every bar-magnet there will be a self-demagnetizing action the value of which, at the middle of the bar, depends, for a given intensity of magnetization, on the length of the bar relatively to its cross-section, on the permeability of its parts, and on the distribution of its surface-magnetism. Owing to the circumstance that with every kind of steel the permeability is neither constant, nor stands in any simple or even single-valued relation to the flux-density; any calculation of the actual polar distribution for rods or bars is exceedingly complicated and indeed impracticable.

As is well-known, the one and only form of magnet that is practicable for calculation is that of the ellipsoid, the properties of which are that for any and every value of the permeability, and when placed in any uniform field, the surface magnetism is so distributed that the magnetic force which this distribution of polarity exerts in the interior is uniform at every point within. Hence the internal demagnetizing force everywhere within is constant; the

resultant field at every point of the interior (if the structure is homogeneous and isotropic) is also constant, and the internal flux-density cannot but be uniform.

Du Bois and others have determined by experiment the demagnetizing actions of cylindrical rods of various dimensions, and have compared them with ellipsoids of revolution of similar dimensional proportions.

In the case of ellipsoids, it is natural to compare the value of the intensity of the self-demagnetizing force with the value of the internal magnetization $\mathscr{I}$, because both of these are uniform throughout the interior. For an ellipsoid of revolution of given axial proportions, whether highly or only slightly magnetized, both $\mathscr{H}_d$ the self-demagnetizing force, and $\mathscr{I}$, are proportional to one another. By definition $\mathscr{I}$ is the quotient of the magnetic moment by the volume. For a given size of equatoreal cross-section of the prolate ellipsoid, the magnetic moment and the volume are both proportional to the axial length. But for ellipsoids of given equatoreal section and of different lengths, the self-demagnetizing force $\mathscr{H}_d$ (for a given $\mathscr{I}$, or a given m) does not follow any simple function of the axial length. For small changes of length it is nearly proportional to the inverse square of the axial length, but is accurately expressible only in terms deducible from a rather troublesome elliptic integral. Maxwell and Du Bois (following F. Neumann) have given the general formulæ. But because both $\mathscr{H}_d$ and $\mathscr{I}$ are for an ellipsoid of given ellipticity proportional to one another, it was quite natural to regard the quotient of the former by the latter—that is to say the amount of self-demagnetizing force per unit of intrinsic magnetization—as a sort of natural coefficient, and to recognize it as a *self-demagnetizing factor*. Du Bois (following Maxwell) assigns to it the symbol $\mathcal{N}$. It has a definite value for ellipsoids of revolution of any assigned ellipticity. Thus for an ellipsoid of equatoreal diameter 1 and axial length 10, the value of $\mathcal{N}$ is 0·2549 whatever the degree of magnetization. Thus if an ellipsoid of this form be magnetized so that $\mathscr{I}$ has the value 100 C.G.S. units, the self-demagnetizing force within the ellipsoid will everywhere have the value of 25·49 gauss. Denoting the dimension-ratio of axial length l to equatoreal diameter d by the symbol $\mathfrak{m} = l \div d$ (in Du Bois' notation), then $\mathfrak{m}^2\mathcal{N}$

varies from 25·49, when $\mathfrak{m} = 10$, to 80 when $\mathfrak{m} = 1000$. (See Du Bois, *The Magnetic Circuit*, p. 41.)

But, if we now compare the case of the ellipsoid with that of the cylindrical bar, we find that the matter is not so simple. For with the bar, as stated above, $\mathscr{H}_d$ is by no means uniform throughout the interior, neither is $\mathscr{I}$. The former has its minimum at the middle point of the axis, while the latter has its maximum at the equatoreal section of the bar. To compute the value of $\mathscr{H}_d$ at the middle point (or at any other) is impossible without knowing the law of surface distribution, and this depends on too many conditions to be of service. But the nett value of $\mathscr{H}_d$ for the entire bar can be easily determined by comparing the $\mathscr{B}$-$\mathscr{H}$ curve of the bar (found by experiment) with the $\mathscr{B}$-$\mathscr{H}$ curve of a ring (or infinitely long rod) of the same iron, and taking the difference of the values of $\mathscr{H}$ for some assigned value of $\mathscr{B}$. On the other hand, values of $\mathscr{I}$ can be found by experiment, either magnetometrically, giving the mean value, or ballistically, giving either maximum or mean according as whether the exploring coil on the bar is wound over its whole length or over its equatoreal zone only. The ratio $\mathscr{H}_d \div \mathscr{I}$ so deduced may still be called the *self-demagnetizing factor*, and values found for rods of different dimension-ratios.

Magnets of other forms, for example the slit toroid, or anchor-ring with a gap in it, and the horse-shoe magnet with parallel limbs of given proportions, will likewise have self-demagnetizing factors of their own, dependent on their geometry and on the distribution of their polarities. With them also, neither $\mathscr{H}_d$ nor $\mathscr{I}$ will have constant values at all points within the substance of the magnet; and for each form therefore the term "self-demagnetizing factor" bears a significance different from that which it possesses for the ellipsoid of revolution or for the cylindrical bar.

All previous writers have defined the term *dimension-ratio* as applied to a bar as the ratio between its length l and the diameter d of its circular section. But when we come to deal with forms of cross-section other than circular, it is inconvenient to use this mode of expression. For if we were dealing with a flat bar of breadth b, the curve for self-demagnetizing factors in terms of the ratio $l \div b$ would not be comparable with those for cylindrical bars in terms of $l \div d$.

The preferable way, when such comparison has to be made, is to state a dimension-ratio, for bars of all and every form of section, in terms of the ratio which is borne by the length to the square-root of the area of section. The ratio $l \div \sqrt{A}$, we accordingly propose to denote by the symbol λ. For any given bar we have the relation $\lambda = \mathfrak{m} \times 1 \cdot 128$.

Part II.—Experimental. On the Values of the Self-demagnetizing Factor for Bar-magnets of circular section.

Several investigators, including Ewing, Fromme, Holz, and Ascoli, have written on the factor of self-demagnetization of cylindrical bar-magnets, and have given experimental values for bars having different ratios of length to diameter. The best-known results are those published by Du Bois, who has compared the values obtained with those for ellipsoids having similar axial ratios. More recently, Riborg Mann obtained a series of values slightly higher than those obtained by Du Bois, who has accepted them as more correct than his own figures. Ewing's observations ranged over rods the lengths of which varied from 300 diameters down to 50 diameters. Du Bois' results go from a dimension-ratio of 1000 down to one of 10; those of Riborg Mann from one of 300 down to 5. The magnitude of the outstanding discrepancies may be indicated by stating the values found by different observers for the self-demagnetizing factor N for cylinders having a dimension-ratio of 50. For rods of this proportion Du Bois found $N = 0 \cdot 0162$; Riborg Mann $N = 0 \cdot 01825$. For the ellipsoid of revolution having the same axial ratio of 50, Du Bois and Riborg Mann agree in assigning the value $0 \cdot 0181$, and presumably the true value for the cylinder is less than that figure. Greater discrepancy is found for shorter cylinders. For a dimension-ratio of 10 Du Bois gives $N = 0 \cdot 2160$, while Riborg Mann gives $N = 0 \cdot 25500$.

To clear up, if possible, such discrepancies a research was undertaken in the laboratory at the Technical College, Finsbury.

The bars used were cut from two long rods of best Swedish iron carefully annealed, and for comparison a ring was forged from the same material. To each and all of the rods the

same diameter was given, namely, 1·128 cm., in order that each might have a cross-section of precisely 1 sq. cm. After being turned down to approximate size they were annealed, and then finally turned to the precise size required.

The magnetizing coil used to magnetize the rods was a long coil wound on a brass tube 91·4 cm. in length and 4·75 cm. in external diameter. It was carefully overwound with 5800 turns of wire of No. 20 S.W.G., in seven layers. With this coil a very uniform field could be produced of any desired intensity up to $\mathcal{H}=255$. The uniformity of the field between the ends of this coil was tested by means of a short coil of somewhat smaller diameter, wound on a turned bobbin of hard fibre, of a size fitted to slide inside the brass tube. The wires of this smaller coil were connected with a ballistic galvanometer, the throw of which was observed when the current in the long magnetizing coil was reversed. The field was found to be sensibly uniform for a length of 60 cm.; while the longest specimen of iron was only 40 cm. There was therefore no need to apply any corrections for non-uniformity of field.

The ballistic method was also used for determining the magnetization of the bars. On the middle of each bar was wound an exploring coil of 10 turns of very fine wire, the breadth of each such coil not exceeding 0·25 cm. The galvanometer was calibrated by the short coil previously mentioned, its dimensions being accurately known. The magnetizing current was measured by a standard commercial amperemeter, the readings of which were calibrated at regular intervals of time by a Crompton potentiometer.

Each specimen was mounted on a carrier by means of which it could be inserted centrally in the middle of the long magnetizing coil. The galvanometer calibration having been effected, a test was made of each bar by subjecting it to a series of reversals in fields varying from $\mathcal{H}=20$ to $\mathcal{H}=255$, the throws of the galvanometer being noted; and for each bar a $\mathcal{B}$-$\mathcal{H}$ curve was then plotted.

A similar curve having been plotted from the tests made on the ring, the values of the demagnetizing intensity of field $\mathcal{H}_d$, due to the self-demagnetizing action of the poles of each bar, could then be calculated, for any value of $\mathcal{B}$, by taking the abscissa, corresponding to that ordinate, in the

curve for that bar, and subtracting the corresponding abscissa in the curve for the ring.

Let the field due to self-demagnetization at the mid-point of any bar, for any given flux-density $\mathscr{B}$, be called $\mathscr{H}_d$. Let the total impressed field due to the magnetizing coil be called $\mathscr{H}$; and let the impressed field required in the ring to produce the same given value of $\mathscr{B}$ be called $\mathscr{H}_r$. Then

$$\mathscr{H}_d = \mathscr{H} - \mathscr{H}_r.$$

Then since, by definition, the self-demagnetizing factor N has the value

$$N = \mathscr{H}_d \div \mathscr{I},$$

and

$$\mathscr{I} = \frac{\mathscr{B} - \mathscr{H}}{4\pi},$$

we get

$$N = \frac{4\pi \mathscr{H}_d}{\mathscr{B} - \mathscr{H}}.$$

Fig. 1 (Pl. XXVII.) gives the $\mathscr{B}-\mathscr{H}$ curves for our rods, the dimension-ratios of which varied from 35·6 to 2·66. These curves were sensibly straight lines up to $\mathscr{B}=12{,}000$, or as high as the curves could be carried. The value $\mathscr{B}=10{,}000$ was chosen for the calculation of the self-demagnetizing force and deduction of the self-demagnetizing factor, except for the very short rods in which lesser values of $\mathscr{B}$ were alone available.

Fig. 2 gives as the final result the curve exhibiting the values of the self-demagnetizing factors found, for rods of different lengths, the corresponding values found by Du Bois and by Riborg Mann being added for comparison.

It will be seen (1) that our values are throughout lower than those found by either of these experimenters; (2) that we have carried the determinations down to shorter rods than those examined by either of them; (3) that the discrepancies between their results and ours are smaller as the dimension-ratios are larger.

The fact that our values are throughout lower than those of Du Bois and Riborg Mann is doubtless due to the circumstance that they used a magnetometric method, whilst we have returned to the ballistic method of Ewing. The values of $\mathscr{I}$ which they employ are the mean values deduced from the magnetic moment, and are presumably mean values throughout the length of the bar, whilst our values of $\mathscr{I}$ are

TABLE I.—Demagnetizing Factors for Cylindrical Bars.

$\frac{l}{d}$.	$\frac{l}{\sqrt{\text{Area}}}$.	Demagnetizing Factors.			
		Cylinder.			Ellipsoid of Revolution.
		Du Bois.	Riborg Mann.	Thompson & Moss.	
2·66	3·0	...	...	1·2	
3·55	4·0	...	...	0·83	
4·44	5·0	...	...	0·618	
5·0	5·64	...	0·6800	*0·53*	
5·34	6·0	...	...	0·483	
6·66	7·5	...	...	0·3518	
8·86	10·0	...	...	0·233	
10·0	11·28	0·216	0·2550	*0·198*	0·2549
10·67	12·0	...	...	0·18	
13·3	15·0	...	...	0·1287	
15·0	16·92	0·1206	0·1400	*0·108*	0·135
17·72	20·0	...	...	0·0826	
20·0	22·56	0·0775	0·08975	*0·069*	0·0848
25·0	28·2	0·0533	0·06278	*0·049*	0·0579
26·6	30·0	...	...	0·0438	
30·0	33·84	0·0393	0·04604	*0·036*	0·0432
35·6	40·0	...	...	0·0255	
40·0	45·12	0·0238	0·02744	*0·0223*	0·0266

N.B.—Figures in italics are values got by interpolation.

the values deduced from the action of an exploring coil wound round the equator of each bar, and presumably measure the maximum value of $\mathcal{I}$. As the self-demagnetizing action of a bar depends on neither the mean value, nor the maximum value of $\mathcal{I}$, as we have seen, but on a mean that is impossible to calculate unless the actual surface distribution of the magnetism is known, it appeared to us preferable to take the value of $\mathcal{I}$ that can be ascertained with precision at the place where the self-demagnetizing force has its minimum, namely the centre of the bar.

One point of criticism on Riborg Mann's results may be permitted us. To give us confidence in our results, we have throughout used substantial bars of 1·128 cm. in diameter, and have raised the lengths. Riborg Mann used a single cylinder of iron 11·850 cm. in length, originally of a diameter 1·526 cm., therefore of a dimension-ratio of 7·76. This he turned down successively to smaller and smaller diameters until he reached a diameter of 0·237 cm., giving a dimension-ratio of 50. How he contrived to turn so thin a wire is

remarkable. It would have a sectional area of only 0·0561 sq. cm. Further, while his cylinder was 11·850 cm. in length, his magnetizing coil was only 30 centimetres long and 4 cm. in diameter. The ends of his rod were therefore at points only $2\frac{1}{4}$ diameters distant from open ends of the coil, where therefore the value of the field would differ by some $2\frac{1}{2}$ per cent. from the value of the uniform field at the middle of the coil.

Part III.—Experimental. On the Values of the Self-Demagnetizing Factor for Bar-magnets of rectangular cross-sections of various proportions.

We are not aware that any previous investigator has determined the self-demagnetizing factor for square bars or flat bars of rectangular section such as are often used in magnetic work.

A priori we should expect the self-demagnetizing factors to be less than for bars of equal section of circular form and equal length; since the greater perimeter of the rectangular forms is magnetically equivalent to giving to the end parts a polar expansion, reducing the reluctance of the air-paths of the external magnetic flux, and so bettering the magnetic circuit. And such has proved to be the case.

The experiments were made in exactly the same manner as those for the bars of circular section. Rectangular rods of the softest Swedish iron of various proportions were procured, and reduced by milling-cutter to the required form, so as in every case to have a sectional area of 1 square centimetre; the ratios of breadth to thickness being respectively 1 : 1 ; 2 : 1 ; 4 : 1 ; 6 : 1, and 10 : 1. From each of these rectangular rods pieces were cut of lengths of 10, 8, 6, 5, 4, and 3 centimetres respectively. In all 35 different ones were examined. For each of these a $\mathcal{B}$-$\mathcal{H}$ curve was plotted; and the self-demagnetizing-factors were deduced as before.

In figs. 3, 4, 5, 6, 7, and 8 these various curves are plotted; and in fig. 9 the final results are summed up by plotting the several demagnetizing-factors as functions of λ the ratio of the length to the square-root of the area of section.

Table II. gives numerically the values of the self-demagnetizing factors obtained for various ratios of breadth b to

thickness t of the cross-section, and also for various values of λ. The individual bars were carefully gauged for breadth and thickness, and the slight discrepancies (never exceeding 1 per cent. of the intended ratio) were allowed for; but being small they occasioned no difference in the plotting.

TABLE II.—Demagnetizing Factors for Bars of different lengths and equal Sectional Area, having Rectangular Sections from 10 : 1 to 1 : 1.

$\frac{l}{\sqrt{A}}=3.$		$\frac{l}{\sqrt{A}}=4.$		$\frac{l}{\sqrt{A}}=5.$		$\frac{l}{\sqrt{A}}=6.$		$\frac{l}{\sqrt{A}}=8.$		$\frac{l}{\sqrt{A}}=10.$	
b/t.	N.	b/t.	N.	b/t.	N.	b/t.	N.	b/t.	N.	b/t.	N.
10·03/1	0·828	10/1	0·586	10·05/1	0·44	10/1	0·354	10·2/1	0·2358	10/1	0·17
5·95/1	0·925	5·96/1	0·66	5·99/1	0·488	6/1	0·3885	5·96/1	0·264	5·99/1	0·19
3·98/1	1·02	3·99/1	0·726	3·98/1	0·528	3·96/1	0·415	4/1	0·28	3·98/1	0·20
2·0 /1	1·098	2/1	0·775	2/1	0·575	2/1	0·448	2/1	0·3	2/1	0·22
1·492/1	1·13	1·5/1	0·7980	1·49/1	0·59	1·5/1	0·4645	1·5/1	0·3075	1·495/1	0·22
0·9914/1	1·13	0·99/1	0·8	0·993/1	0·59	0·996/1	0·465	...	...	0·993/1	0·22

It will be noticed that the Table records values also for bars having the ratio of 1·5 : 1; but no curves are given for this ratio, as they were practically the same as those for square bars. In plotting the $\mathcal{B}$-$\mathcal{H}$ curves for these particular bars, it was possible in two cases only to distinguish the curves from those for the square bars, and in these two cases the difference was extremely small.

For equal values of the ratio of l to $\sqrt{A}$, it was found in general that the self-demagnetizing factor, for bars having a sectional ratio of 2 to 1, was about 93 per cent. of that for bars of square section; while for flat bars, having a sectional ratio of 10 to 1, the value of the self-demagnetizing factor went down to about 75 per cent. of that for bars of square section.

DISCUSSION.

Mr. S. SKINNER said the Authors had referred to the ageing of their magnets, but in the actual experiments the field was applied and the measurement taken at once before any ageing could take place.

Prof. F. T. TROUTON pointed out that the self-demagnetizing factor became less as the bar became thinner, and asked if this was likely to continue until the bar became a thin tape.

Mr. A. CAMPBELL asked if experiments had been made on different materials. The curves obtained for soft iron might be very different from those obtained for hard steel. With regard to the standard magnet referred to by Prof. Thompson, he had used one at the National Physical Laboratory and found it a satisfactory way of calibrating ballistic galvanometers.

Mr. RAYNER asked if any experiments had been made on hollow magnets, such as those used in the Kew pattern unifilar magnetometer.

Prof. C. H. LEES asked how the iron ring used in their experiments had been prepared, and suggested that it might be advisable to make use of two or three rings.

Mr. R. S. WHIPPLE said the question of the preparation of a standard magnet was an important one. Had the Authors made any experiments with ball-ended magnets, especially those with thin wires and large balls?

The SECRETARY read a letter from Dr. A. Russell, stating that the paper was a notable contribution to the practical theory of magnetism. The Authors deserved thanks for their lucid comments on the self-demagnetizing factor and for the important experimental results they had obtained. He asked how they had calculated the magnetizing force. Had they assumed that the magnetizing solenoid was infinitely long and that the radius of the circular axis of the ring was infinite? The errors introduced by these assumptions were small, and Dr. Russell indicated how they might be calculated. It could be shown that if the length of the axis of a helical current was greater than six times its diameter, the magnetic flux entering or leaving the end planes was within about one per cent. of half the total flux through the central plane. Seeing that the external field produced by a cylindrical bar-magnet was almost equivalent, except near the ends, to that produced by a helical current, he would not be surprised if the flux leaving the end of a cylindrical bar-magnet was appreciably less than half the total flux through the central plane. He should be grateful if Dr. Thompson could give any data on this point. Further researches on the demagnetizing effect of the free magnetism in the air-gap of a split toroid would be of great interest, and would throw some needed light on the magnitude of the errors involved in the ordinary magnetic formulæ used in engineering.

Prof. S. P. THOMPSON, in reply to Mr. Skinner, said that in their experiments there was an immediate ageing of their magnets on the application of the field. With regard to Mr. Campbell's remarks, different materials would of course give different curves. A forged ring was used in the research, but the forging could not have had much effect on the curves obtained. They had not tested hollow magnets, but experiments had been made on magnets with holes in the ends and the results were similar to those obtained for solid magnets. He had not performed experiments with Robison ball-ended magnets, but the effect of magnetic material at the ends of a magnet was to reduce its self-demagnetizing factor. Referring to Dr. Russell's remarks, he said the corrections he mentioned could be neglected.

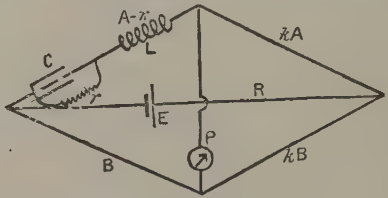
XLII. *On Pirani's Method of Measuring the Self-Inductance of a Coil.* By E. C. Snow, *B.A.**

In Pirani's method of measuring the self-inductance of a coil, it is assumed that the self-inductance of the galvanometer used may be neglected. That this assumption is legitimate in the case in which the discharge of the condenser employed (which, it will be shown, is of the same nature as the discharge through the galvanometer) is continuous has been proved †.

The object of the present analysis is to investigate the case in which the discharge of the condenser is oscillatory.

A condenser of capacity C, the "internal" resistance of which can be neglected, is placed in series with the coil whose self-inductance is to be measured. The condenser is shunted by a non-inductive resistance r. This combination forms one arm of a Wheatstone's bridge, the resistances of the other arms of which are adjusted so that a steady balance holds between the four arms.

One or other of two methods can now be used: (i.) keeping r constant, C can be varied until there is no throw on closing the galvanometer-key before the battery-key; or (ii.) keeping C constant r can be varied, the total resistance of the arm being kept constant, until the same result is arrived at.



The following symbols will be used :—

I, the current in the battery.

x the current in the galvanometer or (in the case of an alternating current) the telephone.

a, the current in the coil whose self-inductance is required.

* Read March 12, 1909.

† O. de A. Silva, *L'Eclairage Electrique*, vol. l. pp. 113–116.

α, the current in the shunt resistance r.
c, the current flowing into the condenser.
b, the current in the resistance B.
a', the current in the resistance kA.
b', the current in the resistance kB.
R, the resistance of the battery.
ρ, the resistance of the galvanometer or telephone.
L, the self-inductance of the coil.
L$'$, the self-inductance of the galvanometer or telephone.
E, the applied E.M.F., not necessarily constant.

The resistances of the arms of the bridge are adjusted as indicated.

The following seven equations are derived immediately from Kirchhoff's laws applied to the various circuits:—

$$a = C + \alpha \quad . \quad . \quad . \quad (1) \qquad b = b' - x \quad . \quad . \quad . \quad (2)$$

$$a = a' + x \quad . \quad . \quad . \quad (3) \qquad I = a + b \quad . \quad . \quad . \quad (4)$$

$$RI + Bb + kBb' = E. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$L\frac{da}{dt} + r\alpha + (A - r)a + \rho x + L'\frac{dx}{dt} - Bb = 0. \quad . \quad . \quad (6)$$

$$\rho x + kBb' - kAa' + L'\frac{dx}{dt} = 0. \quad . \quad . \quad . \quad (7)$$

Also

$$c = Cr\frac{d\alpha}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (1), (2), (3), (4), and (7) we find that

$$a = \frac{fx + kBE}{h} + p\frac{dx}{dt},$$

and

$$b = \frac{kAE - gx}{h} - q\frac{dx}{dt},$$

where

$$f = \{\rho + kB + kA\}\{R + B + kB\} - k^2B^2.$$

$$g = R\{\rho + kA + kB\} + k^2AB.$$

$$h = kA\{R + B + kB\} + kRB.$$

$$ph = L'\{R + B + kB\}.$$

$$qh = L'R.$$

Substituting these values of a and b in (6) we have

$$pL\frac{d^2x}{dt^2}+\left(L\frac{f}{h}+M\right)\frac{dx}{dt}+\left(m-\frac{f}{h}r\right)x$$
$$=-r\alpha+\frac{kBr}{h}E-\frac{kBL}{h}\frac{dE}{dt}, \quad . \quad . \quad (9)$$

where

$$M=L'+Ap+Bq-pr,$$

and $$hm=Af+Bg+\rho h.$$

Differentiating (9), and eliminating α and $\frac{d\alpha}{dt}$ between the resulting equation together with (8) and (9), we finally obtain

$$LCp\frac{d^3x}{dt^3}+\left\{MC+LC\frac{f}{h}+\frac{Lp}{r}\right\}\frac{d^2x}{dt^2}$$
$$+\left\{p+mC+\frac{M}{r}+\frac{f}{hr}(L-Cr^2)\right\}\frac{dx}{dt}+\frac{m}{r}x$$
$$=\frac{kB}{hr}(Cr^2-L)\frac{dE}{dt}-\frac{kBCL}{h}\frac{d^2E}{dt^2}. \quad . \quad . \quad (10)$$

The solution of this differential equation will consist of two parts, a complementary function and a particular integral. The former is the solution obtained by putting the right-hand side of (10) equal to zero. This part will, consequently, be the complete solution in the case for which E is constant.

In this case the value of x is, therefore,

$$x=A_1e^{-\alpha_1 t}+A_2e^{-\alpha_2 t}+A_3e^{-\alpha_3 t},$$

where α_1, α_2, and α_3 are the roots of the cubic equation

$$LCpy^3-\left\{MC+LC\frac{f}{h}+L\frac{p}{r}\right\}y^2$$
$$+\left\{p+mC+\frac{M}{r}+\frac{f}{hr}(L-Cr^2)\right\}y-\frac{m}{r}=0,$$

and A_1, A_2, and A_3 are constants determined by the initial conditions.

From the above value of x—the current through the galvanometer—the value of α, the current through the shunted resistance, can be obtained.

For, from (1) and (8) above, we have

$$\mathrm{C}r\frac{d\alpha}{dt}+\alpha=a=\frac{f}{h}x+\frac{k\mathrm{BE}}{h}+p\frac{dx}{dt}.$$

Putting in the values of x and $\frac{dx}{dt}$, and solving for α, we find

$$\alpha=\mathrm{A}_5e^{-\frac{t}{\mathrm{C}r}}+\frac{k\mathrm{BE}}{h}+\mathrm{A}_1\left(\frac{f}{h}-p\alpha_1\right)\frac{e^{-\alpha_1 t}}{1-\mathrm{C}r\alpha_1}$$
$$+\mathrm{A}_2\left(\frac{f}{h}-p\alpha_2\right)\frac{e^{-\alpha_2 t}}{1-\mathrm{C}r\alpha_2}+\mathrm{A}_3\left(\frac{f}{h}-p\alpha_3\right)\frac{e^{-\alpha_3 t}}{1-\mathrm{C}r\alpha_3},$$

A_5 being another constant determined by the initial conditions.

From this we see that the discharge of the condenser will be continuous or oscillatory according as α_1, α_2, and α_3 are all real or one real and two imaginary (these are the only cases which have to be considered for a cubic equation). But from the above value of x, if α_1, α_2, and α_3 are all real it is seen that the discharge of the galvanometer is continuous, while if two roots are imaginary it is oscillatory. Hence the discharge of the condenser is of the same nature as that through the galvanometer.

To consider the case of the oscillatory discharge of the condenser, therefore, we must put

$$\alpha_2=k_2+ik_3,$$

and consequently

$$\alpha_3=k_2-ik_3,\qquad i \text{ being } \sqrt{-1}.$$

The terms $\mathrm{A}_2e^{-\alpha_2 t}+\mathrm{A}_3{}^{-\alpha_3 t}$ now become $\mathrm{A}_4e^{-k_2 t}\cos(k_3t-\epsilon)$. A_4 and ϵ being other constants.

The complete value of x now is

$$x=\mathrm{A}_1e^{-\alpha_1 t}+\mathrm{A}_4e^{-k_2 t}\cos(k_3t-\epsilon).$$

The initial conditions will be of the form

$$(x)_0=0,\quad \left(\frac{dx}{dt}\right)_0=u,\quad\text{and}\quad\left(\frac{d^2x}{dt^2}\right)_0=v,$$

u and v being constants.

Differentiating the expression for x and putting $t=0$, we

shall have the following equations to determine the constants A_1, A_4, and ϵ.

$$A_1 + A_4 \cos \epsilon = 0,$$

$$A_1\alpha_1 + A_4 k_2 \cos \epsilon - A_4 k_3 \sin \epsilon = -u,$$

$$A_1\alpha_1^2 + A_4(k_2^2 - k_3^2) \cos \epsilon - 2A_4 k_2 k_3 \sin \epsilon = v.$$

These give

$$\Delta A_1 = v + 2uk_2,$$

$$\Delta A_4 \cos \epsilon = -v - 2uk_2,$$

$$\Delta A_4 \sin \epsilon = \frac{1}{k_3}\left\{(\alpha_1 - k_2)(v + 2uk_2) + \Delta u\right\}.$$

where $\Delta \equiv (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$, α_2 and α_3 having the values used above.

In the case of a constant E.M.F. a ballistic galvanometer is used. This measures the total quantity of electricity flowing from the time $t=0$. The conditions of the experiment require that the galvanometer should give no deflexion after the current has flowed for an appreciable time. This condition* can be expressed by $\int_0^\infty x dt = 0$, the upper limit of integration being taken as infinity, since the current will always be zero after a small interval has elapsed. This condition will give

$$0 = A_1 \int_0^\infty e^{-\alpha_1 t} dt + A_4 \cos \epsilon \int_0^\infty e^{-k_2 t} \cos (k_3 t) \,.\, dt$$

$$+ A_4 \sin \epsilon \int_0^\infty e^{-k_2 t} \sin (k_3 t) \,.\, dt;$$

i.e. $$\frac{A_1}{\alpha_1} + A_4 \cos \epsilon \frac{k_2}{k_2^2 + k_3^2} + A_4 \sin \epsilon \frac{k_3}{k_2^2 + k_3^2} = 0.$$

Also $2k_2 = \alpha_2 + \alpha_3$, and $k_2^2 + k_3^2 = \alpha_2\alpha_3$.

* Lord Rayleigh, B. A. Report, 1883, p. 444, has drawn attention to the imperfections of a galvanometer as an instrument for indicating whether the integral sum of the transient currents through it is zero or not. See also a paper by Alex. Russell, M.A., M.I.E.E., Phil. Mag. 6th series, vol. xii. No. 69, 1906, where a full investigation and explanation of the necessary corrections is given.

Putting in the values of the constants, therefore, we have

$$\{v+u(\alpha_2+\alpha_3)\}\{\alpha_2\alpha_3+\alpha_1(\alpha_1-\alpha_2-\alpha_3)\}+\Delta\alpha_1 u=0;$$

$$i.e.\ \{v+u(\alpha_2+\alpha_3)\}\{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\}+\Delta\alpha_1 u=0,$$

and, therefore, $$v+u(\alpha_1+\alpha_2+\alpha_3)=0. \quad . \quad . \quad . \quad . \quad (11)$$

Now from equation (9) above we have, when $t=0$,

$$p\mathrm{L}\left(\frac{d^2x}{dt^2}\right)_0+\left(\mathrm{L}\frac{f}{h}+\mathrm{M}\right)\left(\frac{dx}{dt}\right)_0=\frac{k\mathrm{B}r}{h}\mathrm{E},$$

the term $\frac{d\mathrm{E}}{dt}$ being always zero in the case of a constant E.M.F.

Also, since $(a)_0=0$, we shall have, on putting $t=0$ in the expression for a, $\frac{k\mathrm{BE}}{h}+p\left(\frac{dx}{dt}\right)_0=0$.

Hence

$$p\mathrm{L}\left(\frac{d^2x}{dt^2}\right)_0+\left\{\mathrm{L}\frac{f}{h}+\mathrm{M}+pr\right\}\left(\frac{dx}{dt}\right)_0=0.$$

$$i.e.\ \ p\mathrm{L}v+\left\{\mathrm{L}\frac{f}{h}+\mathrm{M}+pr\right\}u=0.$$

Also

$$\alpha_1+\alpha_2+\alpha_3=\left\{\mathrm{MC}+\mathrm{LC}\frac{f}{h}+\mathrm{L}\frac{p}{r}\right\}\div\mathrm{LC}p.$$

Using these last expressions, (11) becomes

$$\mathrm{C}\left\{\mathrm{L}\frac{f}{h}+\mathrm{M}+pr\right\}-\left\{\mathrm{MC}+\mathrm{LC}\frac{f}{h}+\mathrm{L}\frac{p}{r}\right\}=0.$$

$$i.e.\ \ \frac{p}{r}(\mathrm{L}-\mathrm{C}r^2)=0, \quad \text{or} \quad \mathrm{L}=\mathrm{C}r^2.$$

Therefore, in the case of the oscillatory discharge of the condenser, with a constant E.M.F., the relation $\mathrm{L}=\mathrm{C}r^2$ holds. Hence the expression for the inductance of the coil is independent of the inductance of the galvanometer.

Cass Institute, E.C.,
Feb. 23, 1909.

Discussion.

Mr. A. Campbell expressed interest in the Author's thorough investigation in the first part of the paper. In the case of alternating currents, however, Mr. Snow's final result involved the resistance and inductance of the galvanometer circuit. Since the condition for a balance was that the current through the galvanometer should be zero at every instant,

the final result could not involve either of these quantities, both could be altered to any extent without affecting the balance. The correct solution for alternating currents could, however, be obtained without difficulty. Thus, let P be the effective resistance of the coil L, and let $\sigma = A/B$, the rest of the notation remaining unchanged.

Then $$P + L\omega\sqrt{-1} + \frac{r}{1 + rC\omega\sqrt{-1}} = \sigma B.$$

Separating the real and imaginary parts,

$$L = rC(\sigma B - P) \quad \text{and} \quad rLC\omega^2 = P - \sigma B + r.$$

Hence $$L = \frac{r^2 C}{1 + r^2 C^2 \omega^2} \quad \text{and} \quad P = \sigma B - \frac{r}{1 + r^2 C^2 \omega^2}.$$

Thus it is seen that the balance depends on the frequency, which must therefore be accurately known in order to determine L and P. Mr. T. Smith had worked out the problem by the Author's method, and had found the source of the error. With alternating currents, the initial assumption that the resistances of the four arms are A, B, kA, kB was impossible. When this was put right, B_1 and B_2 both could be equal to zero, which conditions easily reduced to the equations given above.

Dr. Russell pointed out that M. Silva began the investigation of the accuracy of Pirani's method two years ago. He had not, however, discussed the case when two of the roots of the cubic were imaginary. He congratulated the Author on having completed Silva's investigation, and said that his solution of the cubic would be helpful in other problems.

The Author, in reply, thanked Mr. Campbell for his remarks and said he would make the necessary alterations in the part of the paper dealing with alternating currents.

XLIII. *A High-Potential Primary Battery.* *By* Wm. S. Tucker, *B.Sc., A.R.C.Sc.**

[Plate XXVIII.]

This battery may be used for maintaining at known high potentials such conductors as the needle of the quadrant electrometer, for charging condensers in capacity and insulation tests, for calibrating electrometers and electrostatic voltmeters, and for producing constant electrostatic fields.

It is composed of a large number of elements in series. These elements consist of carbon and pure zinc with a saturated or nearly saturated solution of calcium chloride

* Read March 12, 1909.

as electrolyte. The carbon is in the form of graphite used for blacklead pencils, and is about 5 cm. long. A strip of zinc foil about 5 mm. wide is bound to one extremity of the carbon by thin copper wire as in the diagram, and the strip is bent so that it nearly makes contact with the carbon of the next element. This contact, however, is just prevented by paraffin wax. The connected ends are melted into a table of paraffin wax contained in a shallow wooden tray. Elements are mounted in parallel rows, the ends of alternate rows being connected to brass terminals. In the battery illustrated (Pl. XXVIII.) 900 of these elements are arranged in series. The table of wax with the elements is then inverted, and the exposed portion of the elements dipped into a shallow tray containing the solution. On being withdrawn, each carbon-zinc pair holds solution between owing to capillary effect. With ordinary care there is no danger of fracturing the elements, but to protect them they are enclosed in a frame of perforated zinc.

The solution, which is highly hygroscopic, is of such strength that it neither increases nor diminishes in bulk on exposure to the air if the temperature and humidity of the latter do not appreciably change.

The battery is arranged so that a row or any portion of a row of elements can be employed, hence any desired voltage from that of one element to that of all the elements can be obtained.

To effect this, sets of 50 elements corresponding to nearly 50 volts can be introduced by changing the connecting-wire from one brass terminal to the next in the row of terminals fixed to the front of the battery, as seen in the figure.

By means of the plug-key shown at the top of the battery on the right-hand side, a set of 25 more elements may be introduced or cut out—the key occupying for this purpose a gap in the front or at the back of the apparatus. Still another row of 25 elements can be introduced, one element at a time, by means of a slide-contact which passes over a number of brass studs insulated by ebonite, and each connected to an element below. This is also shown in the figure.

It has been found possible to obtain 1·02 volts per element

and to keep this constant to within 0·1 per cent. for about two hours and to with 1 per cent. for half a day. With a suitable solution the battery will act for over a week without repetition of the dipping, and another dipping after that time immediately puts it in working order again.

If the battery is short-circuited for a short time no apparent damage is done, owing to the very high resistance of the element. For the same reason no shock can be felt and no appreciable spark can be obtained when the terminals are momentarily touched. If, however, condensers are placed across the terminals, they are quickly charged, and a strong spark can be obtained.

When the terminals are again insulated the elements gradually recover, owing to the apparent depolarizing action of the porous graphite. It is this effect which distinguishes the battery from others of the same type; thus, with the copper-zinc type there is always a tendency for the voltage to fall, and there is no recovery after the battery is depolarized, without drying the elements.

After use the battery may be cleansed of the exciting fluid by immersing it in the tray, through which a stream of cold water is caused to pass. The battery can then be withdrawn and allowed to dry.

The zinc shows remarkably little corrosion after three or four months of use, and separate experiments have shown that the local action with calcium chloride is very much less than with ammonium chloride. Moreover, there is no sign of creeping, and hence the insulation is maintained.

This insulation is a special feature; all terminals are mounted on ebonite, and all wire connexions inside the battery are covered with paraffin wax.

Since the battery can be dried, it can be stored for an indefinite period, but is always ready for immediate use when required. Moreover, should any element be accidentally broken, the defect is easily located, and a new one can be quickly inserted without disturbing the others.

It may be mentioned that the battery is very compact, the tray being only 18 inches long, 10 inches wide, and 2 inches deep. The materials employed cost comparatively little, and the life of the battery when carefully treated should compare

very favourably with that of others used for the same purpose.

Discussion.

Mr. F. E. Smith congratulated the Author upon his battery, which he said was small, cheap, and effective. He asked if Mr. Tucker had made experiments with other elements, and whether he had constructed a single cell of low resistance and investigated the change of potential with time.

Mr. A. Campbell asked what happened if the battery was short-circuited, and what determined whether "creeping" would occur when using any particular solution.

Mr. S. G. Starling said one of the important points about the battery was that it could be washed and put away, and was then ready for use on any future occasion.

The Author, in reply, said that the only other elements he had used were copper and zinc. He had used different liquids, but always got a lower voltage. Short-circuiting of course caused a fall of potential, but there was not much chemical action on account of the very high resistance of the battery.

XLIV. *The Effect of Radiations on the Brush-Discharge.* *By* A. E. Garrett, *B.Sc.**

The effect of the radiations from radium on the electric discharge produced by means of a Wimshurst machine carrying three pairs of plates each 2 feet diameter, between two brass spheres of 27 mm. and 48 mm. diameter respectively, has been studied by Willows and Peck †. They found that the sensitivity of the discharge to the action of radium radiations rapidly increased with increase of spark-length; the extinction of the visible discharge was accompanied by a decrease in the current; the radium was most effective when held opposite the positive end of the spark-gap; the effect of the radiations from radium could be stopped by the action of a magnet; Röntgen rays did not produce any apparent effect upon the discharge; and when the anode was made sufficiently small (no. 22 wire was used) the discharge passed almost entirely in the brush form, and the radium had no apparent effect.

* Read March 12, 1909.

† Phil. Mag. March 1905, pp. 378 *et seq.*

It was with the idea of ascertaining how far the above results apply when the discharge is produced by an induction-coil, that the present experiments were undertaken. The experiments are not completed, and there is no opportunity to do so at present; it was, however, thought that the results so far obtained would be of sufficient interest to warrant their publication. The induction-coil used was a 6-inch coil, and the primary current was so regulated, and the coil so arranged, that the discharge took the form of a positive brush, and a luminous glow on the cathode. The radium used in all the experiments about to be described was 5 milligrams of strong radium bromide contained in a capsule and covered with mica.

When using the coil it was always found that the radium had to be placed much nearer, in order to produce any visible effect upon the discharge, than when using the Wimshurst.

When the anode of the coil was a copper wire 3 mms. diameter and the spark-gap was 6·3 cms., the radium had no apparent effect on the discharge whether the end of the wire was flat or carefully rounded.

Some experiments were tried with a second spark-gap in parallel with the first, the knobs of the second being just so far apart that no visible discharge took place between them when a steady discharge occurred across the first gap. When the radium was so placed that the visible discharge in the first gap was extinguished, sparking at once took place across the second gap, so that in this case radium produced an effect equivalent to an increase of the resistance of the first gap. When the gap was small the radium was unable to extinguish the discharge; the smallest gap on which any effect was observed was one of 2·5 cms. Willows and Peck (*loc. cit.*) found that radium produced little effect when the knobs of the Wimshurst were less than 3–4 cms. apart.

As the sensitive nature of the brush was found to depend upon the shape and size of the anode, it was thought probable it might also depend upon the material of which the anode was made. Experiments to test this were next tried.

Effect of Material of which the Knobs are made.

Equal-sized anodes were made of copper, brass, zinc, iron, and carbon. In each case a rod 9 mms. diameter with hemispherically turned end was used.

In the case of the first three no difference in the distance at which radium was effective could be observed. When iron was used the radium had to be placed much nearer; and when carbon was used it was practically impossible to stop the brush. This was tried with spark-gaps of 5 cms., 10 cms., and 11·5 cms., respectively. The cathode was first a zinc ball 3·75 cms. diameter, and then a brass plate, and similar results to the above were obtained in all the experiments.

The brass anode was next carefully covered with soot, and it was found that this made it much more difficult to extinguish the brush. When, however, the soot was scratched off a small spot on the rounded end, the brush became just as sensitive as with clean brass. It was also found, as a general rule, that the radium was more effective if the anode was kept carefully polished; in fact with short spark-gaps it was often impossible to extinguish the brush unless this was done. Again, when a small capacity was added to the machine the brush became more sensitive. This would lead one to imagine that the sensitivity of the brush depends upon the oscillatory nature of the discharge, and that the latter depends upon the nature of the anode.

The nature of the material of which the cathode was made was not found to have any effect upon the kind of brush produced. The above experiments were repeated with the Wimshurst machine, with similar results; and since it was found that a much more sensitive brush could be obtained by this means, the machine was used in the subsequent experiments.

With reference to the peculiar action of the iron, and more particularly the carbon positive electrodes, we know that the cathode and anode falls of potential * depend, among other things, upon the metal of which the electrodes are

* Thomson, 'Conduction through Gases,' pp. 442 & 458.

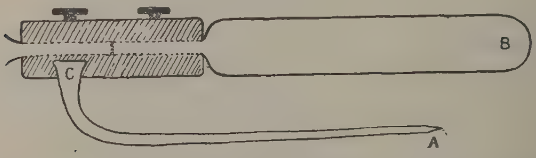
made; unfortunately, however, owing to the difficulties connected with absorbed gases, no such data have been obtained for carbon electrodes. On the other hand, the pressure in the air-gap * is known to be about twice as great for carbon as for the metals used; and this of course would make it more difficult for side influences to upset the discharge after it has once been established.

The oscillatory nature of the sensitive brush was next investigated.

Discontinuous Nature of the Discharge.

The brush was examined by means of a rotating mirror which was driven at a uniform rate by a small motor, and the relative periods of the discharges were ascertained by counting the number of brushes visible in the mirror. By this means it was at once discovered that the degree to which the brush is sensitive to the action of radium depends upon the discontinuity of the discharge. In order that the plates of the Wimshurst might be rotated at a uniform rate, they were driven by a motor; but still it was frequently found impossible to produce a sensitive brush at will. On this account the following arrangement, which was also useful for comparing a steady brush with one which was very sensitive to the action of radium, was used.

Diagram I.



B is the anode, A a piece of thick copper wire pointed at A and soldered to the connexion at C. When A was at the correct distance from end B, a sensitive brush appeared at B, and a brush on which radium had no effect at A.

In the rotating mirror the reflexion of A appeared as an almost continuous band of light, while only a few images of B (3 to 7 according to the anode) were seen; and when the

* Thomson, 'Conduction through Gases,' p. 394.

radium was held sufficiently near, the images of the brush B disappeared, while that of A continued unaltered. Positive knobs of different substances were tested, and it was found that the period of the brush-discharge with carbon as anode was about one half that when copper was used. Again, the images of the carbon brush were evenly spaced, but those formed when zinc or copper were used appeared in little groups.

If the gap was gradually increased when using carbon as anode, the brush ultimately became slightly sensitive, and at the same time the period appeared to change, the number of images seen in the mirror decreased in the proportion 7 : 5. The spacing of these images was also much less regular, and they showed a tendency to arrange themselves in groups.

Iron anodes behaved in the same way as carbon, but the brush was more sensitive than with carbon, although less so than with copper and zinc. [Ordinary iron rod was used, so the effects observed may be due to the carbon contained.]

Since the brush was more sensitive when the images appeared in groups, it was thought possible to see by means of a low-power microscope some difference in the appearances of sensitive and non-sensitive brushes.

Microscopic Appearance of the Brush.

For this purpose a microscope fitted with a 3-inch objective was used. A sensitive brush when viewed with the naked eye appears as in diagram II (a); but with the microscope the stem B is found to be built up of several small stems (6–10). When radium is gradually brought near the brush, these stems are seen to become fewer in number, until only one is left; then if the radium is pushed still nearer, the brush is entirely extinguished. With the non-sensitive brush as produced at the pointed anode, the stems B as seen through the microscope appear thicker, and seem to arise from small mounds as in diagram II (b), which to the naked eye appear as a glow on the anode. This was also visible when the rounded carbon anode was used, but the mounds were then not so conspicuous. When a fairly sensitive brush was examined, and the electrodes were gradually separated so as to produce

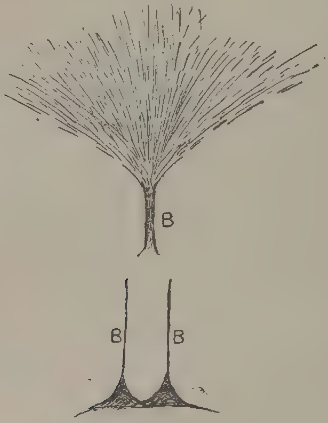
a more sensitive brush, the stems B were found to diminish in brightness and at the same time become fewer in number.

Hence it would seem that the sensitive brush is due to the intermittent nature of the discharge, and that the period of the discharge as ascertained by the spacing of the images is

Diagram II.

II (*a*).

II (*b*).



an irregular one; while the non-sensitive brush at the carbon anode is produced by a discharge which not only has a shorter period, but also one of a more regular nature. The non-sensitive brush from a pointed anode formed a practically continuous band of light in the rotating mirror.

Before concluding this part of the experiments the effect produced by a magnetic field was tried.

Effect of Magnetic Field.

In this connexion I was able to confirm the observation of Willows and Peck (*loc. cit.*); for as soon as the field was applied, the brush, which had been previously stopped by radium, started again vigorously, thus showing that it is the β radiations of radium to which the effect is due. The magnetic field produced no visible effect on the brush, when tried under exactly similar conditions without the radium. This experiment was tried with the induction-coil.

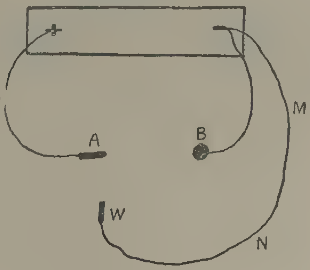
Cook * and Bowlker † have experimented with side attachments, and have found that these have a marked effect upon the nature of the discharge ; so it was thought that an investigation in that direction might throw some light upon the manner in which the β radiations effect the brush-discharge.

Experiments with Side Attachments.

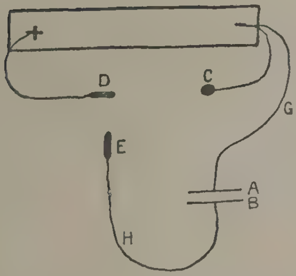
A. *With Induction-Coil.*

Diagram III.

III (*a*).



III (*b*).



M N W and G A B H E are bare copper wire.
A and B in III (*b*) are brass plates 9 cms. square.

Using III (*a*).—When M N W was insulated and attached to the cathode the brush obtained at A was much brighter, more clearly defined, and more constant. The brush so obtained was generally much less sensitive than when W was absent. It was somewhat easier to obtain a good brush when

* Phil. Mag. 1899, vol. xlvii. pp. 40 *et seq.*
† Phil. Mag. 1904, vol. viii. pp. 897 *et seq.*

a plane surface was used for the cathode than when a ball was used. The best brush was obtained when W was held facing the anode at a distance a little too great for sparks to pass between W and the anode.

Using III (*b*).—G A B H E was insulated and attached to the cathode. When E pointed to the anode in a direction at right angles to the gap, and was gradually moved towards it, the brush at D suddenly disappeared. If the distance A to B was altered, the distance of E from D had also to be altered. When A was nearer to B, E had to be placed farther from D, and *vice versâ,* in order that the brush at D might just be extinguished. On the whole, the brush was not so steady with this arrangement, except when E was just on the point of extinguishing it. In III (*b*) the relative position of A B with respect to the spark-gap D C had no influence upon the results obtained. If brushing just occurred from A to B, the brush at D was most easily extinguished by E. When sparking took place between A and B, the brush at D was by no means so sensitive to the action of E. When the brush at D was extinguished by E, it could be made to reappear by placing a sheet of ebonite, cardboard, or even paper, between A and B or D and E. If A was replaced by a point the same phenomena could be observed. When A, B, and E were in almost the correct positions to extinguish the brush at D, it was found to be much more easily affected by the radium than usual; in other words, the action of radium and the side attachment E is additive; for if E was in almost the correct position to extinguish the brush at D and the radium was gradually brought near, the brush was extinguished, but reappeared on moving either the radium or E further from D.

If G A B was attached to the anode it was difficult to obtain a steady brush, and the brush when obtained was not so sensitive to the action of radium as when G A B was attached to the cathode.

B. *With the Wimshurst Machine* the same kind of results were obtained; the following in addition were also noted. When W (III *a*) was attached to the anode and was of the same size and material as the latter, but the wire M N was coiled, W could be placed much nearer B than A could, and

yet a brush occurred at A and no visible discharge at W. When W was pointed and attached to the cathode, if placed facing the anode the brush was extinguished, and no visible discharge occurred at W; but the brush reappeared when any solid body was held between W and A. When a brass ball was used instead of a pointed rod, the brush at A was not extinguished. If the anode was earthed through a high resistance, a steady brush was obtained, and at the same time the machine was not able to reverse. The degree to which the brush was sensitive could be altered by changing the resistance through which the anode was earthed. When the earth was too good it was not possible to work the machine. With the anode so earthed the radium had to be placed on an insulated stand, since it was found that under these conditions radium was more effective if held in the hand; and it was even found possible to stop the brush by placing the hand alone, within 10 cms. of the anode. As a rule, the brush produced when the anode was earthed was far less sensitive to insulated radium than usual. It was generally found that Röntgen rays had no effect on the brush unless the tube was placed very near the anode. Is it possible that the effect produced by the radium &c. may be due to a non-luminous side discharge taking place between the radium &c. and the anode? To test this point an attempt was made to ascertain whether a current between the radium and the anode could be detected, but the results obtained were very irregular.

Willows and Peck (*loc. cit.*) found, however, that a decrease took place in the current across the gap when the brush was extinguished.

A sensitive brush produced by the Wimshurst could not be stopped by a side discharge from an induction-coil, and, in fact, it was found that the side discharge must be produced by the same source as the brush itself for the former to be effective, unless the side arrangement is near enough for sparking to take place.

The natural conclusions from these experiment seems to be that the brush owes its sensitive nature to the oscillatory nature of the discharge; and that it is possible the action of the radium may be due to side discharges which are set up by its radiations.

In conclusion I desire to thank Dr. R. S. Willows for the kindly interest he has taken throughout the course of this work.

Cass Institute,
Jan. 1909.

Discussion.

The Secretary read a letter from Dr. R. S. Willows stating that the experiments of the Author were the only ones that connected the extinction of the spark with any other feature of the discharge. The most obvious explanation seemed to be that on account of the ionizing action of the rays the potential never rose high enough to cause luminescence. This explanation was apparently ruled out by the fact that Röntgen rays do not cause extinction, although the ionization they produce is much greater than that produced by the radium used. It was thought that the intermittence of the rays was their cause of failure, but it was found that Lenard rays, also produced intermittently, caused extinction. In spite of this difference in the action of Lenard and Röntgen rays, the Author's experiments showing that the discharge in order to be sensitive must be intermittent, render it probable that the cause of the failure of the Röntgen rays to produce extinction must be looked for in their discontinuous character. Neither the Author nor he (Dr. Willows) could obtain a sensitive discharge with a positive point: it was therefore of interest to note that Lebedinsky had recently obtained this by connecting in parallel with a Leyden jar, thus producing the Author's condition of a periodic discharge of fairly long period.

Prof. C. H. Lees expressed his interest in the experiments, and referring to the side discharge to the radium, said that it was possible that a discharge towards the radium might be dispersed by the air without giving an actual current from the radium.

Mr. A. Campbell asked what was the frequency of the oscillation in the sensitive brush.

The Author, in reply to Prof. Lees, said that on several occasions he had detected a slight current from the radium to earth, but the results were so irregular that he omitted them from the paper. With reference to the question of frequency of the discharge producing the sensitive brush, he had compared it with that of a discharge in a hydrogen tube in which the discharge was produced by an induction-coil the primary circuit of which was interrupted by a tuning-fork making 86 $\sim$ per second. The number of images visible in the rotating mirror was in the case of the sensitive brush double that given by the tube, so that the frequency was a very low one.

XLV. *On the Least Moment of Inertia of an Angle-Bar Section.* By H. S. Rowell *.

The determination of the least moment of inertia of an angle-bar section is important in the design of a strut by the Euler-Rankine equations.

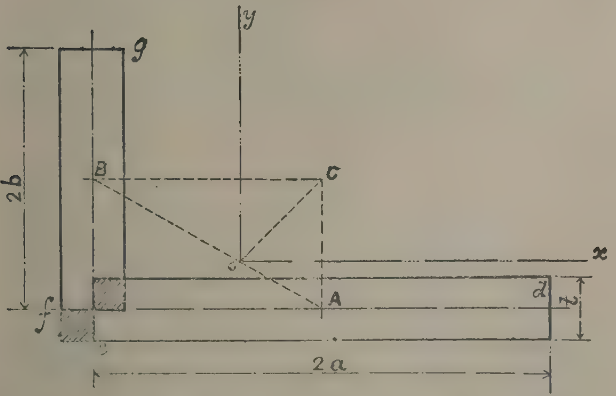
If k be the least radius of gyration of the section, $2a$ and $2b$ the lengths of the legs as shown in fig. 1. Then, very approximately,

$$\frac{k^2}{b^2}=0\cdot17+\frac{6\cdot4\left(\frac{a}{b}-1\right)}{100\left(\frac{a}{b}-1\right)^2+50}. \quad . \quad . \quad . \quad (\mathrm{I})$$

This formula is a very good approximation indeed when $1<\frac{a}{b}<2$. When $4>\frac{a}{b}>2$ the following linear form is better

$$\frac{k^2}{b^2}=0\cdot21-\cdot022\left(\frac{a}{b}-2\right). \quad . \quad . \quad . \quad . \quad (\mathrm{II})$$

Fig. 1



These formulæ were obtained in the following manner:—The principal axes of inertia through the centre of gravity were determined to a good degree of approximation by assuming that the section may be replaced by the two

* Read March 12, 1909.

rectangles *de* and *fg* (see fig. 1). The centre of gravity of these two rectangles may be readily found by the following simple construction. From A and B the centres of the rectangles draw perpendiculars AC and BC (as in fig. 1) meeting in C, from C draw C*o* bisecting the angle ACB (using 45° set square) and meeting AB in *o*. Then *o* is the C of G. If we take coordinate axes *ox* and *oy* through *o* parallel to the legs, we may easily show that the coordinates of A and B relatively to these axes are

$$\left(\frac{ab}{a+b}, -\frac{b^2}{a+b}\right) \text{ and } \left(\frac{-a^2}{a+b}, \frac{ab}{a+b}\right)$$

respectively.

Then the moment of inertia of the system about *ox* is readily found after algebraical reduction to be

$$\alpha=\frac{2t}{3(a+b)}\left\{b^3(4a+b)+\frac{t^2}{4}(a+b)a\right\}. \quad . \quad . \quad (1)$$

By symmetry the moment of inertia about *oy* is

$$\beta=\frac{2t}{3(a+b)}\left\{a^3(a+4b)+\frac{t^2}{4}(a+b)b\right\}. \quad . \quad . \quad (2)$$

The product of inertia of the system about these axes is

$$\phi=-\frac{2a^2b^2t}{a+b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is shown in text-books that if θ be the angle which either principal axis makes with the axis of x, then

$$\tan 2\theta=\frac{2\phi}{\beta-\alpha},$$

whence by substitution from equations (1), (2), and (3)

$$\tan 2\theta=\frac{6}{\frac{b}{a}\left(\frac{b}{a}+4\right)-\frac{a}{b}\left(\frac{a}{b}+4\right)+\frac{t^2}{4ab}\left(\frac{a}{b}-\frac{b}{a}\right)}. \quad . \quad (4)$$

The term in t^2 may be neglected for the proportions commonly used in practice. Moreover, it will be observed that a small error in the determination of θ is unimportant since by definition in the neighbourhood of a principal axis

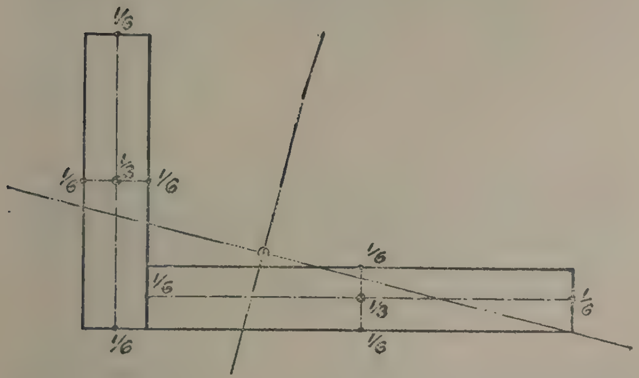
$$\frac{dI_1}{d\theta}=\frac{dI_2}{d\theta}=0,$$

I_1 and I_2 being the principal moments of inertia.

The remainder of the work was chiefly done by graphical methods. Sections were taken from the standard forms of manufacturers, the centre of gravity was formed by the construction above given, a correction being made for the two small squares shown shaded in fig. 1. This correction amounts to $\frac{0 \cdot 088\, t^2}{a+b}$ in the direction C*o* remote from C and is very small. The principal axes were then drawn in according to the values found from equation (4).

The section was divided into the two rectangles shown in fig. 2, and each rectangle replaced by its equimomental

Fig. 2.



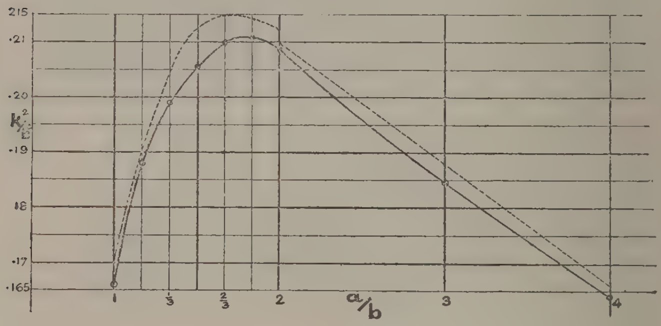
system of particles, viz., $\frac{1}{6}$ of its area at mid-point of each side and $\frac{1}{3}$ of area at centre. Perpendiculars were drawn from each particle to the axis of least inertia, and the square of each one multiplied into the mass of its respective particle. In this way it was found that the moment of inertia of the actual section was greater than that of the centre lines (both being of equal mass) by about 5 or 6 per cent. The least radius of gyration of the centre lines was then determined for various values of $\frac{a}{b}$ ranging from 1 to 4.

The values of $\frac{k^2}{b^2}$ so found were then plotted against $\frac{a}{b}$ (this is seen to be a rational proceeding from considerations

of similitude), and the full-line curve of fig. 3 was thus obtained.

The dotted-line curve shown in the same figure is that given by the empirical formulæ I. and II., and while allowing somewhat for the thickness of the section, it always errs on the side of safety from about one to two or three per cent.

Fig. 3.



The dimensions of angles are generally given as the lengths of the outside edges. The least radius of gyration can easily be expressed in terms of these. Let a_1 and b_1 be the lengths of the outside edges and t the thickness of section, then

$$a=\frac{a_1}{2}-\frac{t}{4}, \quad b=\frac{b_1}{2}-\frac{t}{4}.$$

Putting these values in equation (I) we have, assuming $b_1=6t$, a more handy practical form, viz.

$$\frac{k^2}{b_1^{\,2}}=\cdot 037+\frac{1\cdot 35\left(\frac{a_1}{b_1}-1\right)}{100\left(\frac{a_1}{b_1}-1\right)^2+50}.$$

The greatest value of $\frac{a_1}{b_1}$ in the British Standard Sections $=2\frac{1}{2}$, and this last equation should not be used where $\frac{a_1}{b_1}$ exceeds this.

There was much tedious work involved in the above computations, which would not have been completed but for the kind encouragement of Professors Perry and Harrison.

Discussion.

Dr. Russell expressed his interest in the problem. He suggested that the phrase "second moment of an area" was preferable to "moment of inertia of an area." He asked the Author whether he had compared the numbers obtained by his formulæ with the numbers given in manufacturers' catalogues. He noticed that manufacturers introduce corrections for tapering flanges, rounded corners, etc., and that the numbers they gave for the minimum radius of gyration varied largely with the thickness of the web. Large factors of safety were used in practice, but it was highly desirable that the theoretical numbers should be as accurate as possible. Unfortunately the Author had made some approximations which were not really necessary, and these made it difficult to know what weight to attach to his results. Personally he thought that the best way of finding the second moments of angle-bar or other sections was to use an Amsler's integrator. These integrators had been extensively used for many years, more particularly by naval architects, and their accuracy was not inferior to that of graphical methods.

XLVI. *A Note on the Production of Steady Electric Oscillations in Closed Circuits and a Method of Testing Radiotelegraphic Receivers.* *By* J. A. Fleming, *M.A., D.Sc., F.R.S., and* G. B. Dyke, *B.Sc.**

In testing radiotelegraphic detectors the difficulty is generally to obtain facilities for working in actual stations and at various distances. Thus, if an inventor desires to know whether an improvement which he has made in oscillation detectors is an advance on anything yet done, he must be able to test this receiver at a station in correspondence with others at various and at considerable distances, and even then quantitative measurements are difficult, or impossible, to obtain on account of the continually varying atmospheric conditions which, as is well known, introduce an element of difficulty in connexion with long-distance radiotelegraphy, or else on account of the limited or continually changing distance between the sending and receiving station.

The first-named author of this paper has therefore been seeking for some years past for a method of testing receivers

* Read March 26, 1909.

within very moderate distances which can afford all the advantages to be obtained by working over long distances without any of the disadvantages.

This has now been achieved by the use of closed electric circuits or magnetic oscillators instead of electric oscillators. In a paper read before the Physical Society on October 25th, 1907 (see Phil. Mag. Dec. 1907 or Proc. Phys. Soc. Lond. vol. xxi. p. 47), "On Magnetic Oscillators as Radiators in Wireless Telegraphy," by J. A. Fleming, the author gave two formulæ : one for the radiation in watts from a linear oscillator of the Hertzian type of length l, and the other from a square closed circuit of area S, on the assumption that the oscillations were persistent oscillations having a root-mean square value a and a frequency N. These formulæ were as follows :—

$W = 87 \times 10^{-20} l^2 a^2 N^2$ (for the open or electric oscillator).

$W = 4 \times 10^{-38} S^2 a^2 N^4$ (for the closed or magnetic oscillator)

In the above formulæ W stands for the radiation in watts, l for the length of the linear oscillator, and S for the area of the closed or magnetic oscillator.

These formulæ show that in the case of the open or electric oscillator the power radiated varies as the square of the current-strength and as the square of the frequency, whereas in the case of the closed or magnetic oscillator it varies as the square of the current, but as the fourth power of the frequency. Hence, for any such frequencies as are used in radiotelegraphy and for such dimensions as are generally possible, an open or linear oscillator has much greater radiative power than a closed oscillator of about the same linear dimensions. Accordingly, if two closed-circuit oscillators are placed at a certain distance apart and oscillations set up in one of them, and the other one used as a receiving-circuit, the current in the receiving-circuit can be made extremely feeble when the oscillators are separated by not more than a few hundred yards, and we can avail ourselves of such a means to provide what is the equivalent to two radiotelegraphic stations with open or linear oscillators separated by many hundreds of miles. The convenience, therefore, of the closed or magnetic oscillator is very great because it is possible to set up in an ordinary building, such

as a College or Technical Institution, two square circuits. both under cover and within a reasonable distance of each other, which can be equivalent to two radiotelegraphic stations separated by several hundred miles.

In order to supply the necessary conditions for quantitative work, one of these closed circuits must be made the seat of perfectly constant oscillations, damped or undamped, by preference damped oscillations. The appliances which have now been in use in the radiotelegraphic research Laboratory at University College, London, for many years past for this purpose are as follows :—

The source of electromotive force may be an induction-coil or transformer. If a transformer, it must then be operated by a current from some supply circuit or from an alternator. If an induction-coil, it is preferably operated by large secondary cells, the primary current being interrupted by some good form of mercury coal-gas brake. We have used for this purpose with great advantage the Béclère mercury coal-gas brake, in which a jet of mercury is raised by a rotary pump and squirted against the copper plate in an atmosphere of coal-gas. The mercury brakes that are generally sold for use with paraffin oil as the insulating medium for the mercury, are very messy in use and give great trouble by necessitating constant purification of the mercury and are by no means constant in action. Any mercury turbine brake, however, can be converted into a coal-gas brake by making the vessel gas-tight and employing, instead of paraffin oil, an atmosphere of coal-gas supplied from a small rubber bag under slight pressure. We have also used with even greater advantage for some time a mercury turbine brake by Schall, thus converted into a coal-gas brake which will work for hours at a time for many months without the slightest attention. The next element in the oscillatory circuit is the spark-gap. To obtain perfectly constant results, it is necessary to cause a jet of air to impinge upon the spark-gap to destroy the arcing which otherwise would take place, and, as shown in another paper, the result of this air-blast is to remove the causes of the irregularity in the discharge current (see the following paper on "The Effect of an Air-Blast upon the Spark-Discharge of a Condenser charged

by an Induction-Coil or Transformer," by J. A. Fleming and H. W. Richardson).

It is also desirable to enclose the spark-balls in a cast-iron chamber for silencing purposes.

The form of spark-gap therefore used by the authors consists of a cast-iron chamber S (see fig. 1) closed by a lid with a glazed peephole in it, the distance of the spark-balls being adjustable by a screw, and a glass jet J being arranged so as to cause a steady jet of air from a small Lennox blower under a pressure of 16 to 20 inches of water, to impinge upon the spark-gap. An aperture is left in the cast-iron chamber, through which this air escapes. By using a small spark-gap not more than

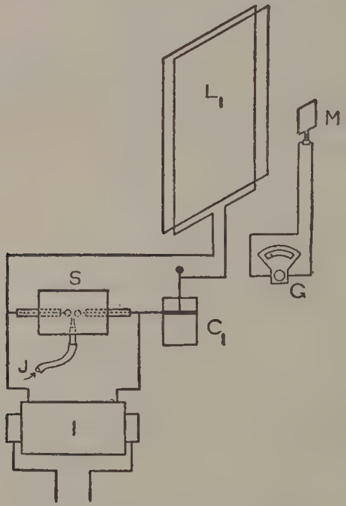


Fig. 1.—Transmitting Circuits.

3 mm. in length, and a suitable pressure of air, it is possible to maintain oscillations of great constancy in a circuit which includes the spark-gap S, a suitable condenser C_1, and the closed circuit L_1 which constitutes the radiator. The radiator is preferably made by winding 8 or 10 turns of stranded insulated wire upon a square wooden frame, which may be anything from 2 feet to 8 or 10 feet in side. The condenser may be an ordinary leyden-jar, or preferably an oil-con-

denser consisting of metal plates placed in anhydrous paraffin oil. The high-frequency capacity of this condenser can then be measured accurately, and also the inductance of the radiative circuit and the frequency of the sparks can be ascertained by a spark-counter, as described previously by one of us. The mean-square value of the current in the oscillatory circuit can be determined by a hot-wire or thermo-electric ammeter inserted in it, or in a circuit M inductively coupled to it, and it will be found that if the above arrangements are adopted, the mean square value of the discharge current in the oscillatory circuit can be kept extremely constant for hours together. Signals can also be automatically sent by interrupting the primary circuit of the transformer or induction-coil by a key operated by a punched tape. In this manner, a succession of Morse signals may be sent, or long and short signals of any kind for the purpose of testing the transmission of any particular words or letters. If the spark-chamber is made of thick cast-iron the apparatus will be nearly noiseless, and may therefore be set up in a Laboratory without disturbing other workers, which is not the case when

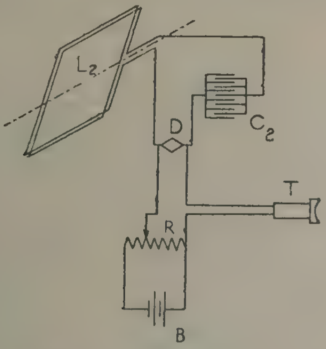


Fig. 2.—Receiving Circuits.

an open oscillatory spark is employed. At a distance, say of 50 to 150 feet or more, another square circuit may be set up consisting of a similar square coil of insulated wire L_2 (see fig. 2) and a condenser C_2, which is preferably a condenser of variable capacity for tuning purposes. A convenient form is one consisting of fixed semicircular plates and a number of

movable semicircular plates fixed on a shaft, which can be rotated, so as to bring the second set of plates more or less in between the first set, the vessel being filled with a highly insulating oil. Two of such closed oscillatory circuits can be set up at a distance, say of 50, 100, or 200 feet within a large building, and even the interposition of brick walls makes no difference, provided they do not contain metal girders.

To test, then, a radiotelegraphic detector of any kind, it is necessary to be certain that the detector *per se*, when unconnected to the oscillatory receiving-circuit, is not directly affected by the spark at the distance at which the sending and receiving circuits are set up; but this can easily be done, and then any particular type of oscillation-detector, D (see fig. 2), whether of the current actuated type or the potential actuated type, can be tested as to sensibility by inserting it, either in series with the condenser of the receiving-circuit, or in parallel with the condenser of the receiving circuit. The detector D is associated with a telephone T and a battery B shunted by resistance R as usual. The use of the closed circuits has this great advantage, that being directive radiators and absorbers, it is possible, by a displacement of the planes of the magnetic oscillators with reference to one another, to obtain a quantitative measure of the sensibility of any given oscillation-detector. Thus, for instance, it is generally possible, but not always, to find a position for the closed circuit of the receiver, such that at a certain distance no effect can be detected in that receiving circuit by any oscillation-detector, however sensitive. We may then call this the zero position. If the receiving circuit is moved out of the zero position by turning it through a certain angle round any axis, it will begin to be affected by the distant transmitting circuit, and a quantitative measure of any oscillation-detector can be obtained by noting the angle through which the receiving circuit must be turned, so that good audible signals may just be obtained. Whether the action of the sending on the receiving circuit is due to true electromagnetic radiation or to ordinary electromagnetic induction seems immaterial. The result in either case is that the receiving circuit is the seat of feeble electrical oscillations and the oscillation detector has to detect these if it can.

Another method is to maintain the receiving-circuit in its position of maximum effect, but to upset the tuning of the receiving-circuit by varying the capacity of the condenser or the inductance of the circuit. This varies the mean-square value of the received current and from the resonance-curve enables us to get the measure of the current or potential-difference which the particular oscillation-detector under test will just not detect. The authors have been employing arrangements of this kind very successfully for a long time past in investigations connected with improvements in the Fleming oscillation-valve.

It has been found convenient to denote the relative telegraphic value of detectors by stating the angle in degrees through which the receiving-coil has to be rotated from the zero position that good audible signals can be obtained on the telephone. If a note is made of the value in amperes or milliamperes of the current in the closed transmitting-circuit, this can always be recovered, and if the spark-length and spark-frequency are the same, we can always be sure that the sending circuit is in a constant and similar condition when comparative tests are made.

The closed receiving-coil is conveniently made by winding silk-covered copper wire, No. 16 S.W.G., on a square

Fig. 3.—Radiotelegraphic Detector Tester.

mahogany frame, which can be revolved on pivots carried on a baseboard which can itself be set at any required angle (see fig. 3). A divided circle and pointer attached to the

frame serve to show the angle through which the frame is rotated. In general appearance it resembles an instrument used in Physical Laboratories under the name of an earth-inductor, for obtaining small induced currents by means of the rotation of a coil in the terrestrial magnetic field.

If a closed transmitting-circuit is set up at a distance, then it is generally possible to find a position for the receiving-coil, such that it will not detect any signals when coupled with a condenser and tuned and associated with a highly sensitive receiver. On turning the coil through a certain angle the signals will be heard. If a very sensitive oscillation-detector is employed, then there may be no position of absolutely null reception, but there will be a position of minimum reception. Thus, for instance, in a certain case, with a coil used at the Pender Electrical Laboratory, some Fleming oscillation-valves of a new type were found to be so sensitive to oscillations, that no position in which the receiving-circuit could be placed was so completely a position of zero mutual induction that these valves, when used with a telephone, gave no signals from a tuned transmitter. Such valves were called zero valves. Others, on the contrary, could not detect signals until the coil had been turned through 5°, 10°, or 20° from the zero or minimum position. A magnetic detector inserted in series with the coil could not detect the signals from the transmitter until the coil was turned through 15°. An electrolytic detector of a particular make required a rotation of 40°, and a carborundum detector required 45° rotation of the coil to give audible signals on the telephone. These measurements are not given as absolute and final measurements of the relative sensibility of all magnetic, electrolytic, or crystal detectors, but merely as examples of the ease with which the sensibility of these special samples of receivers could be tested for order of sensibility. The instrument has proved of great use in a research being conducted now in connexion with improvements in ionized gas radiotelegraphic detectors.

XLVII. *The Effect of an Air-Blast upon the Spark Discharge of a Condenser charged by an Induction Coil or Transformer.*

By J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London, and* H. W. Richardson, *B.Sc.**

When the oscillatory discharge of a condenser is caused to take place across a spark-gap in the usual manner by charging the condenser by means of an induction-coil or transformer, the intermittent spark which takes place between the spark-balls is a complex effect. It consists partly of the true oscillatory discharge of the condenser and partly of an electric arc, unidirectional or alternating, which is superimposed on the true condenser oscillatory spark. If a hot-wire ammeter or other means of measuring the effective or mean-square value of the discharge current is inserted in the condenser circuit, this current will generally be found to be irregular, and if a radiative circuit is coupled to the condenser circuit as in radiotelegraphy, the radiation from it will be found to consist of trains of waves whose initial amplitude is also variable. This irregularity is a source of difficulty in making radiotelegraphic or laboratory measurements of current, decrement, wave-length, &c., when originated by condenser discharges. The reason is that the moment the condenser begins to discharge, and the first so-called pilot spark takes place between the balls, the resistance of the spark-gap falls, and an arc discharge from the induction-coil or transformer commences across the gap. Until this arc is extinguished the condenser cannot again become charged to any high voltage, and the voltage to which it is charged will depend upon the state in which the ball surfaces are left as regards temperature and smoothness, since these are factors in determining the spark potential, and also on the condition of the air-space as regards conductivity. Accordingly, to produce a uniform oscillatory discharge this true arc-discharge must be either prevented or

* Read March 26, 1909.

arrested at once, and the spark between the balls should arise wholly from energy which comes out of the condenser, and not from energy coming directly from the transformer or coil. When moderate power is being employed this arc-discharge can be best annulled by a blast of air thrown on the spark-gap. This has the effect of blowing away the arc, but does not stop the condenser oscillatory discharge. For a long time past the utility of this air-blast in connexion with practical radiotelegraphy has been noted by one of us (J. A. Fleming), but its use in purely scientific measurements is an advantage, as shown in the following paper.

If a large induction-coil has its secondary terminals connected to a pair of spark-balls, or to the outside pair of a series of balls, so arranged that each gap is not more than a millimetre in width, and if a condenser having a capacity say of 0·005 mfd. is connected across the outer balls, then when the coil is in action intermittent sparks pass at the gap. If these sparks are examined in a revolving mirror or photographed on a moving plate, they present themselves as bright images set at fairly equal distances. If a jet of air under a pressure of 16 or 18 inches of water is thrown on the gap by a glass nozzle, the images are then seen to have a ragged tail or aureole which is blown away from the spark, and this tail is generally reddish in colour and easily distinguished from the bright condenser spark. The tail is the image of the arc-discharge superimposed on the oscillatory spark. To determine the effect of this air-blast the following experiment was tried. A 10-inch induction-coil had its secondary circuit connected to brass spark-balls 3 cms. in diameter set with a gap of 1 mm., and the balls were also connected to a rectangular circuit of round copper wire, the diameter of the wire being 0·162 cm. and the sides of the rectangle respectively 142·1 cms. and 34·17 cms. The ordinary or steady resistance of this rectangle is 0·046 ohm and its high frequency resistance to currents of a frequency of the order of $1{\cdot}25 \times 10^6$ is 0·31 ohm. The inductance of this circuit (calculated) is 5012 cms. In series with this circuit was placed a condenser consisting of metal plates immersed

in paraffin oil, the capacity of which was 0·002645 of a microfarad.

In contiguity to the long side of the above rectangle was placed the bar of a Fleming Cymometer, two such instruments being used in the experiments, called respectively No. 2 and No. 3. The cymometer circuit can have two short fine wires of constantan each about 5 cms. long, inserted in it at pleasure, against one of which a bismuth-iron thermo-junction is attached. By passing measured small continuous currents through this fine wire and connecting the ends of the thermo-junction to a low resistance single-pivot Paul galvanometer, the arrangement can be calibrated as a hot-wire ammeter to indicate directly the mean-square value of the oscillations, which when passed through the fine wire cause a certain deflexion of the galvanometer attached to the ends of the thermo-junction. The other fine wire can be inserted as an added resistance in the circuit of the cymometer. The cymometer consists of a circuit including a spiral wire having inductance L and a condenser of capacity C. which can be continuously varied in the same proportion, so that the oscillation constant ($\sqrt{CL}$) of the circuit can be given any value between certain limits. The cymometer was employed to take a resonance curve of the spark circuit by the usual Bjerknes-Drude method, (i.) when the spark-balls were not subjected to the air-blast, and (ii.) when the jet of air was thrown between them.

An extremely steady jet of air for this purpose can be obtained by the use of a small Lennox blower, which is a fan driven by an electric motor taking a current of about 1 ampere from any electric-lamp supply circuit. The resonance curve is obtained by plotting the values of the root mean-square current (a) expressed as a fraction of the maximum current (A) induced in the cymometer circuit by the damped oscillations in the spark-ball circuit corresponding to various values of the natural frequency n or oscillation constant for any setting of the variable inductance and capacity of the cymometer circuit. Then, if A is the maximum value of this secondary current in the cymometer corresponding to a certain natural frequency N or oscillation constant when

resonance between the circuits exists, and if δ_1 and δ_2 are the logarithmic decrements per semiperiod of the oscillations in the spark and cymometer circuits respectively, we have by Bjerknes' formula

$$\delta_1+\delta_2=D=\pi\left(1-\frac{n}{N}\right)\sqrt{\frac{a^2}{A^2-a^2}},$$

provided that such values of n are selected that n does not differ from N by more than say 5 per cent. If then a known resistance R is added to the cymometer circuit, thus increasing its decrement by a known amount δ_2', and a fresh set of observations taken, another resonance curve can be plotted which should lie wholly outside of the first when their maximum ordinates of both curves are taken as unity. From the formula given by Bjerknes we know then that

$$A^2(\delta_1+\delta_2)\,\delta_2=A_1^2(\delta_1+\delta_2+\delta_2')\,(\delta_2+\delta_2'),$$

where A and A_1 are the mean-square values of the maximum or resonance currents in the two cases.

The actual observed quantity which gives us a is the deflexion of the needle of the galvanometer in connexion with the thermocouple pressed against the resistance wire inserted in the cymometer circuit. If there is no air-blast on the balls this deflexion hardly ever remains steady, the needle wanders to and fro over the scale, and the observer can at best but take a mean reading. The result is that the points plotted for the resonance curve do not lie well on a smooth curve, and it is particularly difficult to plot the important part of the curve near the maximum value. This difficulty has been experienced by other observers. Thus, Mr. R. A. Houstoun has recently made a series of measurements of spark resistance and decrement with spark-balls of various metals (see Proc. Roy. Soc. Edin. 1908, vol. xxviii. p. 369, or 'The Electrician,' vol. 62, p. 636); and he remarks that even with great care taken, measurements of logarithmic decrement of oscillation made with the same spark-gap and circuit do not agree well together, the resistance of the spark-gap appears to vary irregularly. Again, Messrs. J. E. Taylor and W. Duddell in some radiotelegraphic measurements made in 1905 (see Journ. Inst. Elect. Eng. vol. 35, p. 321, 1905)

also complained of the difficulty of obtaining good measurements with a spark-transmitter. These difficulties have their origin in the actions taking place in the spark-gap. If, however, a steady blast of air at a suitable pressure is thrown between the spark-balls, this irregularity is greatly reduced provided the spark-gap is not long. The deflexion of the galvanometer becomes greater and much more steady, and accurate observation of the values of the current a corresponding to known values of n becomes much facilitated.

With the above described arrangements a resonance curve can be easily taken as follows:—The jet of air from the Lennox blower conveyed by a rubber pipe ending on a glass nozzle is allowed to play between the spark-balls, and these are connected together by a condenser of known capacity in series with an inductance which has either been measured or predetermined. This inductance is preferably formed of round copper wire and may be rectangular in form, and its high frequency resistance and inductance can then be calculated by known formulæ.

The cymometer is then placed alongside this rectangle so as to have induced oscillations created in it, and by means of the hot-wire ammeter inserted in its circuit the mean-square value a of the cymometer current is taken for various settings of the cymometer circuit, which give it various assigned natural frequencies n. If A and N are the maximum values obtained we then calculate the value of the sum of the decrements of the spark and cymometer circuits from the Bjerknes-Drude formula given above*. Since the resonance curve is not symmetrical with respect to its maximum ordinate, the best mode of procedure is as follows:—Having taken a series of values of the cymometer current a without the extra damping resistance, and those of the current a_1 when the resistance is inserted in the cymometer circuit, and observed the maximum values A and A_1 of the quantities, for various values of n we plot two curves, each to abscissæ n/N, but one having ordinates a/A and the other ordinates a_1/A_1. The latter curve should lie outside the former, but

* See 'The Principles of Electric Wave Telegraphy,' by J. A. Fleming. Longmans & Co., pp. 221–223.

should have the same value for its maximum ordinate, viz., unity. We then draw a number of horizontal lines across the hump of the resonance curves, as shown in fig. 1, with such ordinates that the greatest length intercepted by the

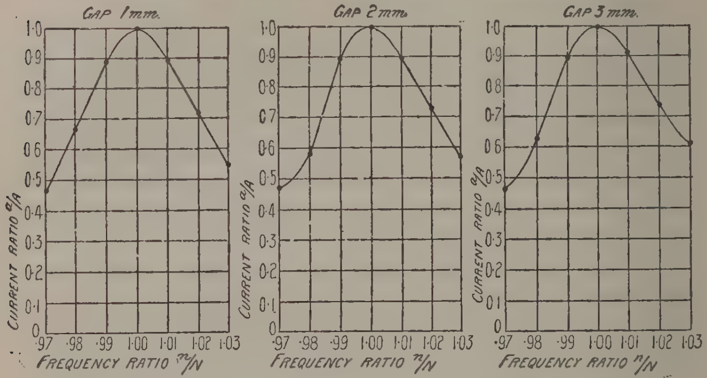


Fig. 1.—Resonance Curves taken with air-blast on spark-gap.

curves is not more than 0·06 on the same scale that N is taken as unity. Let the half length of any such horizontal intercept be called x and the corresponding ordinate a/A be denoted by y, we then calculate the value of

$$\mathrm{D} = \pi x \sqrt{\frac{y^2}{1-y^2}},$$

and this gives us the value of the sum of the decrements of the spark and cymometer circuits. It is well to calulate D from several, say four or five, measurements of x and y for intercepts of different lengths, and then take the mean value for D. In the same way from the outer resonance curve we can calculate the value of D_1. The difference between D and D_1 is the increment in the decrement due to the added resistance wire, and if R is its high frequency resistance and if the frequency of resonance is N and the corresponding inductance of the cymometer is L, we should have

$$\mathrm{D}_1 - \mathrm{D} = \frac{\mathrm{R}}{4\mathrm{NL}} = \delta_2'$$

as a check on the observations.

The following details of one set of experiments will show the advantage gained by the use of the blower. The experiments were made with cymometer No. 2. The primary circuit had an inductance of 5012 cms. and a capacity of 0·002645 mfd., and was charged by a 10 in. induction-coil worked with a coal-gas mercury turbine break. The spark-balls were brass balls 3 cms. in diameter set with spark-gaps of length of 1, 2, and 3 mm. respectively in various experiments. The resistance added in the cymometer circuit was a very fine constantan wire of which the resistance was 7·1 ohms, and a similar wire against which a thermojunction pressed has a resistance of 5 0 ohms.

When the cymometer circuit was adjusted to be in resonance with the primary circuit the resonance frequency N was found to be $1{\cdot}25 \times 10^6$ and the corresponding inductance of the cymometer was 5500 cms. Hence for the added resistance, we have $R = 7{\cdot}1 \times 10^9$ C.G.S., $L = 5500$ cms, and $n = 1{\cdot}25 \times 10^6$. Therefore

$$\frac{R}{4nL} = \frac{7{\cdot}1 \times 10^9}{4 \times 1{\cdot}25 \times 10^6 \times 5500} = 0{\cdot}0258,$$

and for the thermojunction

$$\frac{R}{4nL} = \frac{5 \times 10^9}{4 \times 1{\cdot}25 \times 10^6 \times 5500} = 0{\cdot}0181.$$

The thermojunction ammeter having been calibrated with direct currents so that from the deflexions of the Paul single pivot galvanometer the R.M.S. value of the oscillations passing through the wire which gave any observed deflexion could be obtained, a series of observations was taken by varying the setting of the cymometer slowly and continuously when it was placed in loose inductive coupling with the primary circuit so as to alter its oscillation constant $O = \sqrt{CL}$, or the product of its inductance L and capacity C, from which the natural frequency n of the circuit is at once obtained by the formula $n = 1/2\pi\sqrt{CL}$. At the same time the R.M.S. value of the current a in the cymometer was read off by the thermo-ammeter and the maximum value A also taken and the resonance frequency N. These observations then give the means of calculating D or the sum of the decrements of the primary circuit δ_1 and that of the cymometer δ_2, this last including

that due to the resistance of the hot-wire ammeter. A resonance curve was then drawn and a series of horizontal intercepts measured off near the peak of the curve, and the ordinates of these intercepts also read off on the curve (see fig. 1). The half of the length of the intercept is taken as the mean-value of $1-\frac{n}{N}$, and the corresponding ordinate expressed as a fraction of the maximum ordinate of the curve is taken as a/A, and from this last the function $\pi\sqrt{\frac{a^2}{A^2-a^2}}$ is calculated and the product of these two quantities, viz.,

$$\left(1-\frac{n}{N}\right)\pi\sqrt{\frac{a^2}{A^2-a^2}}$$

gives us $D=\delta_1+\delta_2$, or the sum of the decrements of the primary and secondary circuits.

The resonance curves are plotted to such a scale that the maximum ordinate representing A is unity, and the abscissa of that ordinate representing N is also taken as unity.

The complete set of observations for the 1 mm. spark-gap when subjected to the air-blast was as follows:—

TABLE I.

1 mm. spark-gap with air-blast.

Resonance frequency $= N = 1{\cdot}25\times10^6$.

Oscillation constant of Cymometer $O=\sqrt{CL}$.	Corresponding Frequency of Cymometer circuit $=n$.	Cymometer current in amps. $=a$.	Ratio n/N.	Corresponding actual frequency n.	Corresponding current from Resonance Curve a.	Ratio a/A.
4·15	$1{\cdot}205\times10^6$	0·046	0·96	$1{\cdot}2\times10^6$	0·041 amps.	·343
4·1	1·22 ,,	0·074	0·97	1·211 ,,	0·056 ,,	·469
4·05	1·235 ,,	0·10	0·98	1·224 ,,	0·0803 ,,	·672
4·025	1·242 ,,	0·113	0·99	1·238 ,,	0·106 ,,	·886
4·000	1·250 ,,	0·1195	1·00	1·250 ,,	0·1195 ,,	1·000
3·975	1·257 ,,	0·1146	1·01	1·262 ,,	0·107 ,,	·895
3·95	1·264 ,,	0·1025	1·02	1·275 ,,	0·0865 ,,	·724
3·925	1·272 ,,	0·086	1·03	1·281 ,,	0·0656 ,,	·549
3·9	1·282 ,,	0·077				
3·85	1·295 ,,	0·0558				

The maximum cymometer current = A = 0·1195 ampere without the added resistance in circuit, and the maximum current with the added resistance of 7·1 ohms $= A_1 = 0·0635$.

From the above observations and curve-measurements we calculate the following Table II. :—

Table II.—1 mm. spark-gap with air-blast on gap.

Cymometer Current as per cent. of max. current $= 100\, a/A$.	Value of $\pi\sqrt{\frac{a^2}{A^2-a^2}}$.	Mean Value of $1-\frac{n}{N}$.	Value of $D=\delta_1+\delta_2$.
95	9·58	·0067	·0643
90	6·47	·0098	·0635
85	5·09	·0126	·0642
80	4·18	·0152	·0636
75	3·58	·0177	·0635
70	3·08	·0205	·0632
		Mean Value of D=·0637	

The values of D obtained from the various measurements of the resonance curves are seen to be in very fair agreement. We then calculate the value of δ_2 the cymometer decrement from the formula

$$A^2 D\delta_2 = A_1^2(D+0·0258)(\delta_2+0·0258),$$

where $A = ·1195$, $A_1 = ·0635$, $D = ·0637$, and hence $\delta_2 = ·017$ = decrement of cymometer including that due to the resistance of thermojunction wire.

Hence $\delta_1 = D - \delta_2 = 0637 - ·017 = ·0467$.

Accordingly the total high frequency resistance of the primary circuit $= R = \frac{4NL_1\delta_1}{10^9}$, where L_1 is the inductance $=5012$ cms. and N is the natural frequency $=1·25\times10^6$. Hence we have $R=1·17$ ohms, and since the high frequency copper resistance of the rectangular circuit $=0·31$ ohm, we find that the spark-gap resistance for this 1 mm. spark is 0·86 ohm.

Observations made with 2 mm. and 3 mm. spark-gaps with air-blast in each case gave results of which the final calculations and measurements are embodied in Tables III. and IV.

TABLE III.—2 mm. spark-gap with air-blast.

Cymometer Current as per cent. of max. current 100 a/A.	Value of $\pi\sqrt{\frac{a^2}{A^2-a^2}}$.	Mean Value of $1-\frac{n}{N}$ from Curve.	Calculated Value of $D=\delta_1+\delta_2$.
95	9·58	·0069	·0662
90	6·47	·0010	·0647
85	5·09	·0127	·0647
80	4·18	·0144	·0609
75	3·58	·01695	·0605
70	3·08	·0192	·0595
		Mean Value =	·0627

Maximum Cymometer Current = A = ·101 ampere.
Maximum Cymometer Current
when resistance is added = A_1 = 0·046 ampere.
Calculated Value of δ_1 = ·0443.
" " δ_2 = ·0184.
" " of spark-gap resisistance = 0·80 ohm.

TABLE IV.—3 mm. spark-gap with air-blast.

Cymometer Current as per cent. of max. current 100 a/A.	Value of $\pi\sqrt{\frac{a^2}{A^2-a^2}}$.	Mean Value of $1-n/N$ from Curve.	Calculated Value of $D=\delta_1+\delta_2$.
95	9·58	·0063	·0604
90	6·47	·0095	·0615
85	5·09	·0117	·0595
80	4·18	·0138	·0579
75	3·58	0161	·0577
70	3·08	·0187	·0577
		Mean Value = ·0591	

Maximum Cymometer Current = A = 0·1195 ampere.
Maximum Cymometer Current
with added resistance = A_1 = 0·0652 ampere.
Calculated Value of δ_1 = ·0397.
" , δ_2 = ·0194.
" " of spark-gap resistance = 0·68 ohm.

The same experiments were then repeated with the same spark-gap lengths but without air-blast. It was found to be more difficult to obtain good points for resonance curves, but the final results are as shown in the figures in Table V.

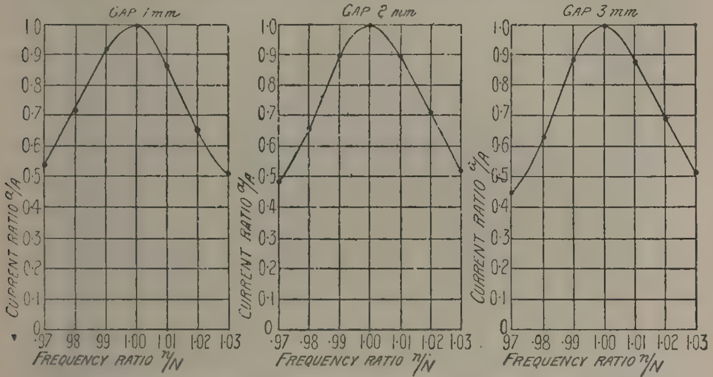


Fig. 2.—Resonance Curves taken without air-blast on spark-gap.

To economise printing we shall write X for the quantity $100\,a/A$, and Y for $\pi\sqrt{\frac{a^2}{A^2-a^2}}$, and Z for the mean value of $1-n/N$.

Then since the values of X and Y are the same in all cases they apply to each spark-length, and need not be repeated if it be understood that the six rows of figures correspond respectively to X=95, 90, 85, 80, 75, and 70 per cent. We accordingly compress the final results for the 1, 2, and 3 mm. spark-gap unblown on in one Table V.

Table V.—1, 2, and 3 mm. spark-gaps without air-blast.

$1-n/N=Z$.			$D=\delta_1+\delta_2$.		
1 mm.	2 mm.	3 mm.	1 mm.	2 mm.	3 mm.
·0072	·008	·0067	·069	·0766	·0642
·00975	·01	·0092	·0631	·0647	·0595
·0017	·0124	·0116	·0595	·0631	·0591
·0141	·0146	·0139	·0592	·0612	·0582
·0164	·0172	·0162	·0587	·0617	·0581
·0187	·0193	·0185	·0576	·0595	·0570
Mean Values			·0613	·0645	·0593

The values of the maximum cymometer current in amperes without the resistance in its circuit ($=A$) and with added resistance ($=A_1$), and the calculated values of the decrements and spark-gap resistances are given in Table VI.

TABLE VI.

1, 2, and 3 mm. spark-gaps without air-blast.

Spark-length in mm.	Max. Cymometer Current in amperes $=A$.	With resistance added $=A_1$.	Primary decrement δ_1.	Secondary decrement δ_2.	Spark resistance in ohms.
1......	·0952	·052	·0423	·019	.75
2......	·1202	·067	·0447	·0198	·81
3......	·119	·066	·0383	·0216	·65

If we collect together the same quantities for the spark-gaps when blown on we have results as in Table VII.

TABLE VII.

1, 2, and 3 mm. spark-gaps with air-blast.

Spark-length in mm.	Max. Cymometer Current $\doteq A$.	With resistance added $=A_1$.	Primary decrement δ_1.	Secondary decrement δ_2.	Spark-resistance in ohms.
1......	·1195	·0635	·0467	·017	·86
2......	·101	·046	·0443	·018	·80
3......	·1195	·0652	·0397	·0194	·68

It will be seen, therefore, that when the air-blast is not applied the various values of $D=\delta_1+\delta_2$ calculated from the resonance curves are more irregular and deviate more from the mean, this being the result of the difficulty of obtaining correct galvanometer readings to delineate a good resonance curve.

The observations also appear to show that the air-blast has not much influence upon the primary decrement or upon the spark-resistance, or at most tends to slightly increase the

spark-resistance for very short sparks. The good effect of the air-blast is seen best when applied to short sparks, which tends to show that it is of assistance in destroying the arcing then occurring. This arc, however, is unable to persist with longer gaps; that is to say, it blows itself out, and accordingly for spark-lengths of 3 mm. or upwards and when using an ordinary 10 in. induction-coil with mercury break as interruptor, the blast is of no special advantage.

Another set of experiments was conducted, the object of which was to ascertain the effect of the air-blast upon multiple and upon very short spark-gaps. At one time some radio-telegraphists were of opinion that an advantage was gained by dividing up a spark-gap into smaller spark-gaps in series. To test this opinion more carefully, a series of six spark-balls were made, each consisting of a pair of brass balls 2 cm. in diameter adjustable as to distance by a screw of 0·5 mm. pitch, having a divided head. These spark-balls were arranged so that they could be put in series with one another, the series forming the spark-gap in an oscillatory circuit having an inductance of 125,000 cms. and a capacity of 0·0045 mfd. A number of glass jets were arranged so that an air-blast could be directed against each pair of spark-balls across the gap. In the oscillatory circuit a hot wire ammeter was placed, so arranged that a reading could be taken of the ammeter first with a single spark-gap of a definite length, and then with a series of 2, 3, 4, 5, and 6 gaps, each adjusted to be equal in length, the sum of them all being equal to that of the single gap employed. Thus, for instance, the effect of a single gap 0·5 mm. could be compared with that of five gaps in series each 0·1 mm., both with the air-blast and without the air-blast against the gaps. As in the case of very short gaps, the phenomenon of multiple sparks exists; that is to say, each interruption of the induction-coil in the primary circuit gives rise not merely to one discharge in the oscillatory circuit, but to a series of discharges, because a discharge takes place between the balls corresponding to the length of the spark-gap employed, and whilst the electromotive force in the secondary circuit of the coil increases and endures, during this time many sparks may take place.

Whether this occurs or not can be determined by means

of a revolving mirror. If the image of an oscillatory discharge produced by an induction-coil is examined in a revolving mirror, then if there is only one spark corresponding to each interruption of the primary coil, a series of widely separated sharp images of the spark will be seen, but if the phenomenon of multiple discharge is taking place, then the images are each seen to consist of 4, 5, 6, or more small sparks rapidly succeeding each other. Also, if there is any sensible arcing at the spark-gap it is at once seen in the image in the revolving mirror in the form of a trail of light of a different colour to that of the true oscillatory discharge-spark. On examining the spark-discharge with and without the air-blast for certain short spark distances, the effect of the air-blast is easily detected, because it is seen to blow away this trail which accompanies the image of the spark-discharge.

In order to avoid misinterpreting the results, it was necessary to be sure that when using say one single spark of 0·5 mm. in length and comparing it with the effect of 5 spark-gaps of 0·1 mm. in series, any increase in the current at the discharge circuit was not due to an alteration in the number of discharges.

As far as could be observed this was not the case when sufficiently short spark-gaps were employed.

The following Table VIII. (p. 679) embodies the results of the measurements. Employing single spark-gaps ranging in length from 0·1 mm. to 3 mm., observations were taken of the mean-square value of the current in the discharge circuit, both with and without the air-blast, and these values are recorded in the first two columns. It will be seen that up to a certain length of gap (about 2 mm. in the case of these experiments) the air-blast had a very decided effect in increasing the mean-square value of the discharge current, but beyond that point it seems to have the effect of diminishing it. On the other hand, if a single gap, say of 0·2 mm., is broken up into two gaps in series each of 0·1 mm. and so on for the other spark-lengths, it is found that dividing the spark-gap into two parts also increases the discharge current up to about 0·8 mm., and after that the current diminishes. If the double air-gap is blown upon, the current is increased

TABLE VIII.

Comparisons of the Effect of breaking up a single Spark-gap into several Spark-gaps in series.

Total Gap.	In 1 Gap.		Separated into 2 Gaps.		Separated into 3 Gaps.		Separated into 4 Gaps.		Separated into 5 Gaps.		Separated into 6 Gaps.	
	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.
mm. ·1	amp. ·54	amp. ·66										
·2	·684	·816	·728	·816								
·3	·628	·912	...	...	·71	·896	...	...	·88	·88		
·4	·628	1·04	·684	1·029	...	...	·88	1·042				
·5	·71	1·095	...	...	...	...	...	...	1·34	1·34		
·6	·755	1·253	·728	1·258	·88	1·142	·937	1·042	1·029	1·029		
·8	·84	1·284	·84	1·228	...	...	·914	1·042	...	...	1·017	1·017
1·0	·896	1·35	·914	1·35	...	...	...	...	·896	·896		
1·2	·975	1·287	·975	1·35	1·042	1·142	·896	·896	...	...	·755	·755
1·5	1·142	1·142	...	...	1·029	1·029						
2·0	1·2	1·2	1·042	1·042	...	...	...	...	·755	·755		
3·0	1·258	1·042	1 017	1·017	·937	·937	...	...	...	...	9·5	9·5

as compared with the same two gaps not blown upon, but is not increased as compared with a single gap blown upon. The same is true when the spark-gap is broken up respectively into 3, 4, 5, and 6 gaps. The effect of dividing the gap up to a certain point is to increase the discharge current, but the effect of the air-blast on the multiple gaps becomes less and less marked in proportion as the number of gaps increases, so that the effect of dividing up a gap, say 0·6 mm., into five gaps of 0·12 mm. each, is to increase the discharge current almost as much as by subjecting the single gap of 0·6 mm. to the air-blast. This increase appears to be due to the suppression of the arc-discharge, as shown by the appearance of the images of the spark in the revolving mirror, and hence the arcing is suppressed by the separation of the gap into multiple gaps, almost as effectively as by blowing upon the single gap. On the other hand, separating the spark-gap into more than five separate gaps seems to result in diminishing the total discharge current beyond a certain very short length of spark. The figures in the table, however, do not show what is observed in practice, namely, the greater steadiness of the discharge current under the operation of the air-blast; and the conclusion, therefore, is that in any experiments in which great constancy is required in the discharge current in the condenser circuit, a great advantage is obtained by using a short spark-gap, and by subjecting the discharge spark to an air-blast, as this both increases the charging voltage of the condenser and steadies the discharge current by abolishing the arc, which would otherwise take place even with an ordinary induction-coil.

A similar set of experiments was tried with a larger alternating current plant and a high tension transformer, constituting one of the wireless telegraph sets of the Radiotelegraphic Laboratory, University College. The alternating current was supplied from a 5-k.w. alternator and raised in pressure by a transformer discharging across a spark-gap between two iron balls, the spark-gap being shunted by a condenser in series with the primary circuit of an oscillation transformer, the secondary circuit of which was inserted between an antenna and an earth connexion, the antenna circuit being

tuned to the condenser circuit. As no suitable hot-wire ammeter was available for measuring the large current in the condenser circuit, readings were taken by connecting the terminals of a hot-wire voltmeter to points on the earth-wire connexion of the antenna, consisting of a copper strip. These points being selected a few inches apart so as to give a convenient reading on the low reading hot-wire voltmeter. These readings, however, are approximately proportional to the currents flowing in the condenser circuit. Spark-gaps were then adjusted from 0·5 mm. to 3 mm. in length, and the reading of the voltmeter taken, both when the spark-gap was blown upon and when it was not blown upon a by a jet of air.

The following table shows the readings:—

Table IX.

Spark Lengths.	Voltmeter Reading.	
	No Air Blast.	With Air Blast.
3·0 mm.	2·4	2·7
2·5 ,,	2·4	2·7
2 ,,	1·85	2·2
1·5 ,,	1·7	1·7
1·0 ,,	1·65	2·0
·5 ,,	·75	1·0
1·0 ,,	2.0	2·25
1·5 ,,	2·1	2·4
2·0 ,,	2·05	2·35
3·0 ,,	2·3	2·45

The Table shows in every case an increase in the current circulating in the condenser circuit when the spark-gap is subjected to the air-blast, but does not indicate the much greater steadiness of the voltmeter-needle which is seen with the air-blast. Without the air-blast the voltage was by no means constant, and the readings given in the first column are therefore a mean reading from which the extreme readings may differ by 10 per cent.

The final result of the experiments is to show that when using spark-gaps, 1, 2, or 3 mm. in width, in a condenser circuit, for the purpose of exciting oscillations, much greater uniformity in the discharge current can be obtained if the spark-gap is subjected to an air-blast as described.

Discussion.

Mr W. Duddell congratulated the Authors and remarked that the papers contained a great deal of useful material. The use of the earth inductor for testing receivers was a feature of the first paper. He asked Dr. Fleming to what extent the effects obtained in the receiving circuit were due to true radiation, and why they had used coal-gas instead of alcohol in their interrupter. Referring to the Authors' method of determining the current in the transmitter, he asked if there was any objection to putting an ammeter in the circuit and reading the current directly. He should also like some more information about the cases in which it was found impossible to obtain a position of the earth inductor giving silence in the telephone. Was it because the distance apart was not sufficiently great or was some physical impossibility involved. With regard to the second paper he had always found it possible to get a uniform discharge by using an alternating current of suitable frequency in the primary. He asked the Authors if it was the arc or the spark which was blown out by the air and whether the part blown out had a spectrum different from the rest.

Dr. W. H. Eccles asked how much of the energy absorbed by the receiver was due to radiation and how much to electromagnetic induction. He had obtained results, depending on electromagnetic induction, similar to those described by the Authors by using very much smaller apparatus, but he had discontinued his experiments because in practice the whole of the energy received was due to true radiation. He pointed out that a receiver adjusted and tested in a laboratory was never in proper adjustment for actual work. He suggested that the reason it was sometimes impossible to get a position of silence arose from stray radiation falling upon the receiver.

Dr. Erskine-Murray pointed out that detectors varied greatly in resistance and that therefore a telephone of suitable resistance should be selected in each test.

Dr. A. Russell thought that Prof. Fleming and Mr. Dyke's method of testing radio-telegraphic receivers would be a great help in judging their relative values. He much appreciated the clear distinction drawn between the function of the spark and the arc as this cleared up some of his difficulties. Although the air-blast of the Lennox blower was doubtless beneficial by preventing arcing, he thought that the fact that the dielectric coefficient of the glass nozzle used was greater than unity might have accelerated the sparking. He mentioned some of the difficulties encountered in computing sparking voltages when there were

two gaps in the circuit. In this case it had to be remembered that the sparking voltages at the moment of the discharge are not equal and opposite. When the potentials of the electrodes are known, however, and they are spherical in shape, the sparking voltages with two gaps in series can be calculated with fair accuracy.

Dr. R. S. Willows pointed out that by blowing out the spark the resistance of the path was increased and the rate of charge of energy thereby altered. The fact that greater regularity and greater energy could be obtained could be easily demonstrated by using an electrodeless discharge-bulb. Referring to the fact that it was necessary to know the self-induction of one of the circuits, he asked the Authors what form of circuit had been chosen and how its self-induction had been calculated. In one of the experiments a resistance r had been added, and he remarked that attention should be directed to its effective resistance under rapidly alternating currents, as this might depend upon whether the added resistance was a pure metal or an alloy.

Mr. L. H. Walter agreed with Dr. Erskine Murray that it was necessary to choose a suitable telephone when making a test. Although the electrolytic detector was supposed to be more sensitive than the magnetic form, it was possible to choose a telephone of such a resistance as to make the magnetic detector appear the more sensitive.

Dr. Fleming in reply said that it was impossible to state precisely what proportion of the current produced in the receiving circuit was due to true radiation from the closed transmitting circuit, and how much was due to electromagnetic radiation, but from his point of view it did not matter. All that was necessary was that a feeble oscillatory current should be produced in the receiving circuit which should be capable of being varied by turning the receiving circuit through a certain angle, and whether this was due to actual detachment of energy from the transmitting circuit, or to the mere movement of lines of magnetic force backwards and forwards through space, seemed immaterial. The oscillat.on detector in any case is a mere detector of oscillations.

In reply to Mr. Duddell, he said that there was no objection to putting an ammeter in the transmitting circuit provided it was a low resistance instrument and did not produce any sensible damping of the oscillations.

As regards the existence of an exact zero point, this seemed to be a question of distance from the transmitter. It had been usual at University College to work with two coils about 60 feet apart, and at that distance some very sensitive oscillation valves detected sounds which might be due to the action directly upon the valve or upon the connecting wires, but by going to larger distance, it was possible to get complete silence at the telephone.

With respect to the use of coal-gas or alcohol, in the interrupter, coal-gas had proved itself to be incontestably superior to alcohol.

As regards the action of the air-blast, it appears tolerably certain that the part of the discharge which is blown away is that due to energy coming directly out of the induction-coil or transformer.

In reply to the remarks of Dr. Eccles, he said that they had found it necessary to work at a certain distance from the transmitter, but that when this was done the order of sensitiveness in which oscillation-detectors were arranged by the apparatus shown was also the order in which they were found to be sensitive when employed in actual radio-telegraphic work. It is necessary not to work the transmitter and receiver too near to one another, otherwise there are direct effects on wire connections, rheostats, etc., which obscure the real effects.

In reply to Dr. Willows, Dr. Fleming said that the reason for choosing the rectangular form of circuit was because the inductance could be readily calculated from formulæ given in well-known text-books.

With respect to the remarks of Mr. Walter and Dr. Erskine Murray, Dr. Fleming agreed that it was necessary to choose a telephone of suitable resistance, and that the results taken with different telephones would not be the same.

XLVIII. *On the Action between Metals and Acids and the Conditions under which Mercury causes Evolution of Hydrogen.* By S. W. J. Smith, *M.A., D.Sc., Lecturer on Physics, Imperial College of Science and Technology* *.

Contents.

§ 1. *Introductory.*—Under ordinary conditions metals like mercury, silver and copper are unable to displace hydrogen from solutions of acids (dilute or concentrated) with appreciable and easily demonstrable evolution of the gas.

An attempt is here made to show how, in the case of

* Read March 26, 1909.

mercury, the fluidity of the metal at the ordinary temperatures may be utilized to exhibit the cause of the inability and to supply a method by which it may be overcome.

The effects with hydrochloric and sulphuric acids only have been examined, but there is no apparent reason why solutions of other acids should fail to yield results of the same kind.

§ 2. *Electrolytic Solution Pressure.*—A method of regarding the question whether a metal will dissolve in an acid solution has been suggested by Nernst and developed by himself, Ostwald and others. The general idea of this method has become familiar, and provides a means of forming a more suggestive picture of what actually happens than is obtained from purely thermochemical considerations (*Cf. e.g.* Thomsen, 'Thermochemistry,' pp. 349 to 356, &c.), unsatisfactory in other ways as well.

In order to present a consecutive account of the present experiments, this view is restated below in a form as far as possible free from hypotheses not absolutely necessary*.

§ 3. *A conception of the Interaction of Metals and Acids.*—Imagine that a metal M comes suddenly into contact with an air-free solution of an acid HX. It is known that in general equilibrium will be impossible; a certain quantity of H will be precipitated upon the surface of M and an equivalent quantity of M will dissolve.

It may be that this interchange will take place only to an indefinitely small amount, but we can safely say that no salt MX is absolutely insoluble and that every metal has at least some tendency to go into solution. It is the relative magnitude of this tendency which is to be regarded as the characteristic variable distinguishing Hg, for example, from metals like Fe and Zn.

On account of the electrostatic resisting forces which would arise, ions of H cannot escape spontaneously from a solution of HX, nor can ions of M escape from the metal M; but with M and HX in contact the conditions alter. The assumed tendencies of M and H to spread beyond their original boundaries can now become effective without

* *Cf.* Nernst, 'Theoretical Chemistry,' 2nd Engl. edit. p. 724.

development of electrostatic resisting forces; because equivalents of H and M can pass across the common surface of acid and metal without change in the electric charge on either side. Neutral molecules of H can escape from solution while ions of M enter.

It is a necessary conclusion from a consideration of this kind (abstract thermodynamics furnishes many other examples) that solution of the metal and precipitation of hydrogen must begin at the interface (assuming no other change possible), whether accompanied by loss or gain of heat from the rest of the system. The familiar parallel is the fact that a gas will always expand spontaneously from higher pressure to lower if the external constraints permit—the work which it does being performed at the expense, if necessary, of its own internal energy.

It would appear, therefore, that every metal must be able to displace to some extent the hydrogen of an acid solution. If in any case the action is imperceptible, it must be because the tendency of the metal to enter solution is very small. The entry of an insignificant amount is then sufficient to balance the tendency. After this, further solution of the metal with escape of hydrogen will be a process like the compressing of one quantity of gas by the expansion of another—the constraints of the two quantities of gas being such that mutual expansion and contraction is the only change possible.

In such a case equilibrium is reached when the work which could be done by any further expansion of the second gas would be less than the work required to increase the compression of the first. Similarly, the replacement of hydrogen by a metal will cease at a point defined by the condition that further escape of hydrogen would produce less available work than would be required to cause the equivalent quantity of metal to enter solution.

§ 4. *Symbolic expression of the argument of* § 3.—Assuming as a rough approximation that the process is reversible and takes place isothermally, the conception may be expressed symbolically as follows.

The work done (diminution of available energy) when one equivalent weight of hydrogen escapes from solution may be

written in the form $\mu^s_h - \mu^n_h$, in which μ^s_h is a function of the strength of the solution (increasing with the concentration), and μ^n_h depends upon the pressure at which the hydrogen escapes (increasing with the pressure),

Similarly the work done (gain of available energy) when an equivalent of the metal enters solution may be expressed as $\mu^s_m - \mu^n_m$, in which μ^s_m is a function of the amount of metal already dissolved per ccm. near the interface, and μ^n_m is a physical constant of M (at the temperature of the interaction and for a given curvature of surface if the metal is fluid), which may be large or small according to the nature of the metal.

The relation necessary for equilibrium is

$$\mu^s_h - \mu^n_h = \mu^s_m - \mu^n_m. \quad . \quad . \quad . \quad . \quad (\text{i.*})$$

The amount of M which enters solution (and of H which escapes) before equilibrium is attained will be determined by the value of

$$c = f(\mu^s_m) = f(\mu^s_h - \mu^n_h + \mu^n_m),$$

where c is the concentration of the salt MX near the surface which is necessary in order that μ^s_m may acquire the value required to satisfy (i.). If no such value of c can arise equilibrium is impossible (unless owing to secondary effects other conditions supervene), and interaction will continue until the supply of acid or of metal is exhausted.

§ 5. *Possible effect of Surface Tension.*—The above statement of the conditions of equilibrium neglects possible variation of the surface energy during the interaction. In many cases there may be relatively little variation; but in one at least, where the metal is mercury and the interface consequently separates two liquids, changes of surface tension

* In the strictly reversible system M : MX : HX : H_{Pt}, the corresponding equation of equilibrium (no electromotive force) is approximately of the form

$$aT \log c_{HX} - E_h = bT \log c_{MX} - E_m,$$

where E_h and E_m are 'electromotive constants' of hydrogen (at given pressure) and the metal, at the temperature of equilibrium T.

are easy to detect. In this case the surface energy can be seen to diminish with increase in the concentration of the mercury salt in solution.

Consequently there is now a greater diminution of available energy than is represented by the left-hand member of (i.), when an equivalent of H is replaced in solution by an equivalent of M. Hence μ^s_m must attain a greater value in the solution before equilibrium is reached than would be required to satisfy (i.).

Thus if $-\gamma$ is the decrease of surface tension which would accompany the exchange of equivalents of H and M across unit surface when equilibrium is attained, we should have

$$\mu^s_h - \mu^n_h + \gamma = \mu^s_m - \mu^n_m, \quad . \quad . \quad . \quad \text{(i. } a\text{)}$$

and the equilibrium concentration of the salt of M in solution would now be

$$c = f(\mu^s_h - \mu^n_h + \mu^n_m + \gamma).$$

§ 6. *The Problem for Experiment.*—If the equation (i. a) represents the condition of equilibrium between mercury and an acid (assuming that the only reaction possible is of the type

$$\mathrm{Hg + HX = HgX + H})$$

it is clear that when the substances come into contact a certain amount of hydrogen must be displaced. Otherwise the right-hand member of the equation can never become equal to the left.

If (as is likely in the case of Hg) the quantity μ^n_m is very small, the quantity $\mu^s_m - \mu^n_m$, (being at constant temperature of the form $k \log c/c_0$, where c_0 is very small), may acquire a considerable value, even when c is small, *i. e.* when only a small quantity of hydrogen has been displaced.

Thus equilibrium may be reached and displacement of hydrogen cease (neglecting diffusion effects) before the amount separated per unit surface has become perceptible.

If, however, some means could be found of removing the mercury salt as fast as it was formed the reaction would continue, and thus the displacement of hydrogen might be rendered evident.

§ 7. *Detrimental effect of Oxygen in the Surface-layer.*—The simplest way of obtaining an experimental answer to the question whether the direct displacement of hydrogen by mercury ever occurs is not immediately obvious. The purest mercury in contact with the air will become coated with a film of condensed oxygen—possibly a minute layer of oxide. Hence, even if the acid with which it may be brought into contact is free from dissolved oxygen, the interaction contemplated in the equations above may be prevented.

When an equivalent of hydrogen forsakes the acid solution in the presence of oxygen, the loss of available energy can be greater than before because, instead of separating as gas, the hydrogen can now become part of a molecule of water. Hence the amount of metal which must dissolve before equilibrium is reached will much exceed that required to give μ^s_m the value sufficient to satisfy (i. *a*) above. The equilibrium value of μ^s_m in the present case may be written

$$\mu^s_m = \mu^s_h - \mu^{OH_2}_h + \gamma + \mu^n_m . \quad . \quad . \quad . \quad \text{(ii.)}$$

In which the quantity $\mu^{OH_2}_h$ is much less than the corresponding term μ^n_h of equation (i. *a*).

It is known from electrical measurements that the earlier stages of this reaction proceed with great rapidity. In the presence of a sufficient quantity of oxygen, μ^s_m may thus almost at once reach a greater value than that required to prevent the evolution of hydrogen*.

It is therefore essential to experiment with a mercury surface as far as possible oxygen-free, or to devise a means of removing from near any mercury surface as much as is desired of the mercury salt in solution.

§ 8. *Possible methods of eliminating this effect.*—Imagine two masses of mercury A and B immersed in the same acid. Let the surface of A be one not originally oxygen-free and surrounded in consequence by solution containing dissolved mercury salt. Suppose that the state of the solution around A has become practically steady without appreciable diffusion of the mercury salt into the region round B. Let the surface

* The effect of the presence of oxygen can also be presented in a thermochemical form, omitted here for the sake of space.

of B be one originally free from oxygen, and suppose that no interaction has yet taken place between it and the acid. Left to itself this mercury might interact with the acid, some of it displacing some of the hydrogen; but suppose that, instead, it is brought into contact with the mass A at one point or more without considerable change in the extent of either surface in contact with the acid.

Well-known electrochemical phenomena leave no doubt as to what will happen. Mercury will immediately begin to deposit on A and to enter the solution round B. Mercury-salt will in fact disappear and appear in equal quantities round A and B respectively until the concentration of the salt in solution round both is the same. This change will be effected by a displacement of anionic 'chains' in the solution from A towards B and by some analogous process (shift of electrons) in the mercury from B to A. The time taken by the process to complete itself will depend upon the length of the ionic chains. If these are short this time will be very small compared with the time taken by A to reach the steady state acquired before the contact.

From the point of view already described this process occurs in a way analogous to the expansion of a gas when the external constraints permit. Here the constraints virtually permit the expansion of the mercury salt from the space round A into the space round B until the concentration is the same in both. The electrical phenomena are incidents, not causes, of the flow of matter which takes place.

From whatever point of view the result may be regarded there is no doubt that metallic contact between A and B would reduce the amount of mercury salt in solution round the former. If B were very large compared with A, the amount of mercury left in solution round A would be very small. Similarly if A were connected with a succession of masses initially like B but small (each being removed before another took its place), the amount of salt in solution round A could be continuously lowered.

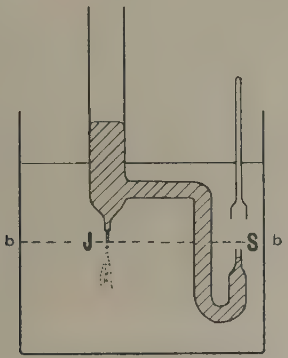
Although it appears possible to obtain oxygen-free surfaces of mercury (like B), they are usually in rapid motion and very difficult to observe. It is easy, however, to study a

surface (like A) subjected to metallic contact with a succession of surfaces much more nearly oxygen-free than itself.

§ 9. *Experimental realization.*—Mercury poured into a vertical glass tube drawn to a fine capillary at the lower end escapes in a narrow stream. Since the surface of the mercury per unit mass is much greater in this than in the tube, a considerable quantity of the mercury below the surface in the tube must enter the surface in the jet. Thus a surface film, upon the mercury originally, must become much thinner in the jet.

Suppose the stream to enter an acid solution as at J in the figure. It will possess much less oxygen per unit surface

Fig. 1.



than mercury at rest. The concentration of oxygen may fall below the amount sufficient to raise μ_m^s to the value required to satisfy (i. a), and therefore a small quantity of hydrogen may form.

The chance of formation of hydrogen will be greatest at the end of the jet where it breaks into drops and suddenly presents a new surface to the solution. If the jet is completely immersed this chance will be only a little greater than at the sides; but it will be much enhanced if the end of the jet just touches the surface of the solution (level of liquid at b in fig. 1).

Thus suppose the length of the completely immersed jet to be l and the velocity of efflux v. The "electrochemical" forces already described will tend to equalize the distribution of mercury salt in solution round the jet, although the various elements of the surface have been in contact with the solution for times varying between zero and l/v. The end of the jet after rupture is a surface bounded by that portion of the rest of the jet which has been longest in contact with the solution, and the concentration of mercury salt round the end will therefore be raised practically instantaneously to a considerable value.

If the jet breaks in the surface of the solution, however, the electrochemical short-circuit is reduced to a minimum because the sides of the jet are now practically out of contact with the solution.

Very little hydrogen can be produced on any element of the surface even when the jet has its greatest efficiency, and the difficulty of formation of extremely small bubbles (owing to surface tension effects) may prevent evolution of gas otherwise possible in accordance with (i. a). Suppose, however, that the jet is connected to a small mercury surface at rest in contact with the solution as at S (fig. 1). If there is less mercury in solution round J than round S, the metal will precipitate at S and enter solution at J. This action will continue as long as the concentration of Hg in solution round S exceeds the practically steady value which would exist round J if S were absent.

Consequently the concentration of the mercury salt round S must tend to diminish continuously until it is as small as that round J. But if, before this can happen, the amount of Hg in solution at S becomes smaller than that required to satisfy (i. a), a new reaction will begin. Direct action between the mercury of S and the acid will occur with evolution of hydrogen and formation of a new supply of mercury salt in solution.

This action will be continuous, for, in virtue of the continuous effect of J, Hg will be continuously removed at S. A steady state (neglecting secondary actions such as described below, § 12) will be reached when the rate of evolution of hydrogen at S is exactly equivalent to the rate of removal of

mercury from solution at S by J. For every equivalent of acid that disappears at S, an equivalent of the mercury salt will appear in solution at J.

§ 10. *Results.*—In the above way it is possible to conceive that mercury will decompose sulphuric or hydrochloric acid with evolution of hydrogen. Experiment justifies the conception, for, if either acid (of sufficient strength) is used in the vessel of fig. 1, a stream of hydrogen is evolved at S.

If the point of the capillary (at J) is below the surface of the acid the effect occurs most rapidly when the head of mercury just suffices to produce a short, approximately cylindrical, jet at J. (The ratio l/v for the jet has then a minimum value because, with increase of head, the jet length increases more rapidly than the velocity.)

The effect occurred with the strongest sulphuric acid available ('pure redistilled' S. G. about 1·84). If the acid is diluted the same evolution of gas can be obtained, but, when the density of the acid falls below about 1·25 the rate of evolution is very slow and occurs only when the jet breaks in the surface of the solution. Finally, when the density of the acid is below about 1·19, there is no perceptible evolution of gas in any position of the jet*.

The experience with hydrochloric acid is similar. With the most concentrated acid used (S. G. about 1·16) there was a fairly rapid evolution of gas when the jet broke in the surface. Gas ceased to come off when the density of the acid fell below about 1·09. The molecular concentrations of these limiting solutions are about the same—roughly 6 equivalent gram mols. per litre—although that of the HCl solution is slightly the smaller †. (*Cf.* § 13 below.)

§ 11. *Proof of Evolution of Hydrogen.*—The gas was not proved to be hydrogen by a direct test in every case; but only in the case most liable to suspicion, viz. when concentrated sulphuric acid was employed; and in one other, viz. when equal volumes of this acid and water were mixed. It

* In practice it is easier to adjust J and S if separate columns of mercury connected by a wire are used instead of the apparatus of fig. 1.

† Mr. J. S. G. Thomas has since made a more exact determination of the limiting concentrations. He finds them to be 6·25 gram equivalents per litre for H_2SO_4 and 5·75 gram equivalents per litre for HCl.

was proved to be hydrogen in the following way :—Collected in a small tube over the concentrated acid, it did not dissolve appreciably in the latter, nor, subsequently, in recently boiled distilled water by which the acid was displaced. It was therefore neither H_2S nor SO_2.

It was difficult to test the gas positively since with the arrangement used it took a considerable time to collect a few cubic millimetres. It was possible, however, to show that it underwent contraction on explosion with oxygen in the following way. A bubble of the gas, 4 mms. long, was collected at the top of a tube which ended in a capillary through which a fine platinum wire had been sealed. About 0·8 mm. of oxygen (prepared electrolytically) was added and a second fine wire was pushed into the collecting tube from below until it reached almost to the first. A spark was then passed, and the remaining gas was found to occupy about 2·5 mm. of the tube *. Thus the gas evolved behaved like hydrogen, the only constituent common to the two acids employed.

§ 12. *Secondary effects with Sulphuric Acid.*—Although, in the absence of oxygen, the simplest direct interaction between mercury and sulphuric acid is

$$Hg_2 + H_2SO_4 = Hg_2SO_4 + H_2, \quad . \quad . \quad . \quad (I.)$$

there are other possible interactions of which the next † in simplicity would be

$$4Hg_2 + 5H_2SO_4 = 4Hg_2SO_4 + H_2S + 4H_2O. \quad . \quad (II.)$$

Here every fifth molecule of the acid may be supposed to be reduced by the hydrogen resulting from the direct action between four molecules of the acid and mercury.

According to the view adopted in this paper the reaction I.

* Mr. W. F. Higgins kindly attempted to make a spectroscopic test of the gas, but various difficulties were encountered which it did not seem profitable to attempt to overcome since the gas had already been proved to be neither H_2S nor SO_2 nor oxygen.

† In the reaction representing the secondary reducing effect of hydrogen, viz.,

$$Hg_2 + (1 + \alpha)H_2SO_4 = Hg_2SO_4 + (H_2 + \alpha H_2SO_4)$$

the minimum value of α is 1/4.

can go on only so long as the concentration of mercury salt in solution does not exceed the value given by (i. a); but according to the same view the reaction II. can occur before and after this limit to reaction I. is passed. It probably does not occur to the exclusion of I. because it involves a greater rearrangement of the constituents of the reacting molecules than is involved in the displacement of hydrogen (*cf.* Thomsen, *l. c.* p. 354 *et passim*).

To take the case of concentrated sulphuric acid, which was carefully examined. The H_2S of reaction II. will interact with a further quantity of the acid precipitating sulphur. It will also in part precipitate the very nearly insoluble sulphide of mercury by interaction with the sulphate in solution. The former reaction may be represented by the equation

$$3Hg_2 + 4H_2SO_4 = 3Hg_2SO_4 + S + 4H_2O. \quad . \quad (II. a)$$

In this case, neglecting for simplicity the energy variation due to the decomposition of the SO_4 ion, the equilibrium value of $\overset{s}{\mu}_m$ may be represented qualitatively by

$$\overset{s}{\mu}_m = \frac{4}{3}\left(\overset{s}{\mu}_h - \overset{H_2O}{\mu}_h\right) + \gamma + \overset{n}{\mu}_m, \quad . \quad . \quad . \quad (iii.)$$

which shows that more mercury must now enter solution before equilibrium is attained than when hydrogen ceases to be evolved in accordance with (i. a). The reaction with precipitation of mercury sulphide leads to a similar result.

Thus we are led to infer that the production of sulphur and of sulphide of mercury may continue after the evolution of hydrogen has ceased.

This inference is fully confirmed by experiment:—

(a) When the diameter of S is small—less or not much greater than that of J—the mercury in solution round S is removed almost at once by the action of J and hydrogen simultaneously appears. Very soon, however, a yellowish-white cloud begins to form (particularly round the portions of S where the curvature is least and where the evolution of hydrogen is most noticeable). This cloud probably arises mainly from the decomposition of H_2S. In fact when the effectiveness of J was decreased by reducing the head some

of the last bubbles to escape seemed (when observed through a microscope) to be surrounded by a film exactly like that which is produced when bubbles of H_2S are passed into concentrated H_2SO_4. The formation of sulphide of mercury can also be detected after the jet has been in action for some time.

These reactions rapidly reduce the rate of evolution of hydrogen at S. The production of sulphur is a process which the action of J cannot reverse and the amount increases continuously, raising the electric resistance of the solution in the capillary and hence diminishing the effectiveness of J. The insoluble sulphide precipitated on the surface of the mercury further increases the circuit resistance. In addition this sulphide seems to prevent the evolution of hydrogen directly although the amount of mercury salt in solution may be very small. For instead of escaping, the hydrogen must now apparently interact with the sulphide in the surface layer producing H_2S and eventually sulphur. Thus after a time the only effect of J at S is to produce sulphur. If, however, the capillary at S is rinsed out by expelling a little of the mercury, the evolution of hydrogen begins again at the fresh surface of acid and mercury.

(*b*) When the diameter of S is large—for example, twenty times that of J—sulphur and sulphide only are formed, however efficient the jet may be. In this case the jet only slowly reduces the amount of mercury salt in solution round S, and the reactions depending on the formation of sulphuretted hydrogen begin before the evolution of hydrogen is possible and continue in such a way that it can never occur. In a qualitative sense it may be said that in this case the slower reaction (requiring greater molecular rearrangements) has time to prevent the first.

It is only when the acid is concentrated that these effects occur, for diluted sulphuric acid is not reduced to or by sulphuretted hydrogen. Thus even in the case (*b*) just mentioned, hydrogen is evolved freely when sulphuric acid solution of S. G. 1·5 is substituted for the concentrated.

§ 13. *Effects of Dilution of the Acids.*—The loss of available energy when an equivalent of hydrogen leaves the acid solution and is evolved as gas at atmospheric pressure falls

continuously* as the solution is diluted. Consequently the amount of mercury which can be in solution round the electrode without preventing the evolution of hydrogen becomes continuously less.

Before the hydrogen can escape it must reach a certain concentration (in the neutral state) in the mercury and in the solution at the surface layer. Disregarding the difficulty of formation of very minute bubbles, evolution just fails to occur when, electrode and solution being saturated, there would be no loss of available energy if an infinitesimally small quantity of hydrogen left the solution and the infinitesimal equivalent of mercury entered.

Thus if there is a limit below which J (however effective) cannot reduce the concentration of the mercury salt round S (§ 9), there is also a limit to the concentration of the acid which can be decomposed at S with evolution of gas. The experiments of § 10 show that this limit is reached at a concentration of about 6 equivalents per litre in the case of each acid.

Further, assuming that equally concentrated solutions of the acids and of their corresponding mercury salts are approximately equally dissociated, one would expect in virtue of equation (i.) that the limiting concentrations of the two acids would be approximately equal. This also agrees with experiment.

It happens, however, that the quantity γ in (i. a) is greater for HCl than for H_2SO_4. It might therefore be anticipated that if the corresponding dissociations were exactly equal, the limiting concentration for HCl would be rather less than for H_2SO_4; but although the results suggest the fulfilment of this anticipation, the data are not sufficiently accurate to

* In the case of concentrated sulphuric acid, dilution seems first to increase the rate of evolution of hydrogen. This result might be anticipated for two reasons. The first effect of dilution is to increase the conductivity and, probably, the ionic concentration of the acid. Thus the effectiveness of the jet and the value of μ_h° will simultaneously rise. Again, some of the hydrogen which would otherwise escape will react with the concentrated acid while in the nascent state (§ 12). There should thus be a particular strength of sulphuric acid for which the rate of evolution of gas is a maximum.

make it worth while to attempt a quantitative proof—especially as there are other possible explanations.

§ 14. *Possible effects at the Jet.*—The explanation which has been given of the behaviour of the still mercury surface S supposes that the concentration of mercury salt round the jet J is small. In fact the behaviour of S is (by hypothesis) controlled by the rate of independent formation of mercury salt at J. Consequently when hydrogen is evolved at S the amount of mercury in solution round J should be insufficient to prevent the evolution of hydrogen at J.

If any such evolution takes place it is very difficult to detect. Mercury was allowed to run for a long time from a capillary tube fused into the top of a glass bulb which was filled with concentrated sulphuric acid and terminated below in a tube dipping into mercury. No trace of gas could be seen. The level of the acid was lowered until the jet broke in the surface, the space above being filled with air. In this case, the first impression is that there is a copious evolution of gas (*cf.* Paschen, *Wied. Ann.* vol. xli. p. 56, 1890). But this is an illusion. The greater part, if not all, of the gas which appears to form at the surface of the mercury drops is air dragged in from above. This effect is very pronounced in liquids of great viscosity. It is less conspicuous in sulphuric acid than it is in glycerine.

In consequence of this phenomenon it is impossible to tell by inspection whether any gas is evolved as the result of chemical action between the acid and the mercury. In a further experiment benzene (which had previously been shaken up with another sample of concentrated H_2SO_4 and then decanted) was substituted for the air above the acid. There was now a continuous circulation of drops of benzene within the acid like that of the air bubbles before. A considerable quantity of gas accumulated at the top of the apparatus; but it may have resulted from some chemical reaction in which the benzene took part. The evolution was even more noticeable when the benzene was replaced by pentane.

It has already been pointed out that the escape of hydrogen at the jet may be impossible although it takes place at S (§ 9). At the latter the effect is cumulative and all the hydrogen

is evolved at the same surface. Here minute bubbles can coalesce into larger ones in which the pressure is not greatly above that of the air. At the jet, however, the hydrogen which mercury replaces must first reach a certain concentration in every fresh element of the jet and of the solution surrounding it. Then it must overcome the resistance to the formation of a minute bubble before it can escape as gas.

That direct action between the jet and the acid might result in visible production of hydrogen, but for the counteractions just described, can be seen by allowing the jet to break in the surface of a concentrated solution of sulphate of copper. The surface of the mercury which collects on the bottom of the vessel containing the sulphate presents a tarnished appearance like that produced by the addition of copper. This result, it will be seen, is in close accord with the present point of view and suggests the need for qualification of the familiar statement : "The more electropositive metals, Cu . . . . . . , precipitate the less electropositive metal Hg." It is probable that other metals, such as Pb and even Cd, could be precipitated by mercury, from solutions of their salts, in a similar way.

§ 15. *A kinetic representation of* §3.—The action between an acid and a metal is formulated in § 3 in a way which avoids the necessity of dealing with the kinetics of the process by which equilibrium is attained. The following is perhaps the simplest picture of what actually happens. The metal is assumed to be monovalent, but it is easy to see what change must be made when the valency is n.

Into the space (the "border layer") which separates acid and metal, Faraday tubes can stretch from the acid and from the metal respectively. Of these, a possible pair forming side by side will be (1) a tube stretching from the acid nearly across the border layer, with its negative end (an anion of the acid) in the solution and its positive end (an ion of hydrogen) near the metal, (2) a tube stretching from the metal, with its negative end on the metal and its positive end (a metal ion) almost reaching the solution.

These tubes forming simultaneously and close together will interact, yielding a molecule of the salt MX in solution and a neutral equivalent of H deposited on M.

The number of interactions of this kind taking place per sq. cm. of surface in the unit of time will depend upon the frequency with which such tubes form side by side, *i.e.*, upon the number of molecules of HX in solution per cc. on the one hand and upon some specific property of the metal (determining the rate at which tubes of the second kind form) on the other. For the "velocity" of the reaction

$$M + HX \longrightarrow H + MX,$$

we may therefore write

$$v = k\, c_h\, C_m,$$

where k is a constant at given temperature, c_h is the concentration of the hydrogen ions in solution, and C_m is a specific constant of the metal M.

As the result of this action, hydrogen will accumulate upon the metal and MX will be formed in solution. A reverse action, similar in kind to the first, now becomes conceivable. This reaction, which may be represented by

$$H + MX \longrightarrow M + HX,$$

will proceed with a velocity

$$v' = k'\, c_m\, C_h,$$

where c_m represents the ionic concentration of the metal in solution and C_h is a specific constant of hydrogen deposited at given pressure upon M.

If a steady state is reached, after a certain quantity of hydrogen has been displaced and the equivalent quantity of M has dissolved, it will be defined by the condition $v = v'$ or

$$k\, c_h / C_h = k' c_m / C_m.$$

We can thus deduce kinetically a result identical with that obtainable by application of the logarithmic formula of Nernst and contained as a particular case in the general equation of § 4.

Since there is no effective transfer of electricity across the border layer, a possible contact difference of potential between metal and solution would not affect the available work equation of § 4. Similarly it would not affect the final equation of equilibrium deduced kinetically. For if a

potential-difference existed it would have the same relative effect upon v' as upon v.

§ 16. *Summary of Conclusions.*—Pure mercury reacts with acid solutions with displacement of hydrogen in the same way as metals like zinc.

The reaction stops before a perceptible quantity of hydrogen is evolved because a very small quantity of mercury salt in solution is sufficient to cause it to cease.

The surface film of mercury which has been in contact with the air probably contains more than enough oxygen to oxidise to water all the hydrogen that would be displaced before the direct action ceased.

The amount of oxygen per sq. cm. of the surface film can be reduced to a very small quantity by allowing the mercury to escape from a containing tube in the form of a narrow jet.

But for certain counteracting influences due to the fact that the substance displaced is a gas, this jet might be used to obtain hydrogen from acid solutions of sufficient strength.

By the aid of the jet the direct action between mercury and the acid, with displacement of hydrogen, can be made continuous. The jet, when in direct communication with a mercury surface at rest in the same solution, prevents the concentration of the mercury salt, formed by the displacement of hydrogen, remaining or becoming large enough at the still surface to stop the evolution of gas. In consequence, hydrogen escapes freely and can be collected and analysed.

Owing to direct action at its own surface, the jet cannot reduce the concentration of mercury salt round the still surface below a certain limit. Further, the amount of mercury salt which is sufficient to stop the direct action at the still surface diminishes with the concentration of the acid solution used.

In consequence it will be impossible to decompose the acid by means of the jet when the strength of the solution falls below a certain limit. This limit might be expected to be about the same for different acids. It was found to be about the same for hydrochloric and sulphuric acids (roughly

6 gram equivalents per litre), although rather lower for the former than for the latter.

The various reactions which occur when metals are placed in contact with concentrated sulphuric acid are elucidated by the experiments described (§ 12).

I am very much indebted to Mr. J. S. G. Thomas, B.Sc., for frequent and valued help while performing the experiments I have described.

Discussion.

Dr. W. Watson congratulated the Author and remarked that in physical chemistry hypotheses were often used which were not based on experimental evidence. Dr. Smith had shown that his hypothesis corresponded with actual physical facts, and his theory could be looked upon with satisfaction.

In reply to a question by Mr. F. E. Smith, the Author explained that the process by which the concentration of the mercury salt in solution was reduced below the amount necessary to prevent evolution of hydrogen was endothermic.

XLIX. *Theory of the Alternate Current Generator. By* Thomas R. Lyle, *M.A., Sc.D., Professor of Natural Philosophy in the University of Melbourne* *.

It has been usual hitherto to ascribe the distortion of the wave-form of the current given by an alternate current-generator to:—

1. "Lack of uniformity and pulsation of the magnetic field, causing a distortion of the induced E.M.F. at open circuit as well as under load."
2. "Pulsation of the reactance causing higher harmonics under load."
3. "Pulsation of the resistance causing higher harmonics under load also" †.

And, as far as I have been able to find out, another cause has been overlooked, namely, the mutual reactions between armature and field, which when the generator is loaded is at least as important as any of the foregoing.

* Read April 23, 1909.

† Steinmetz, 'Alternating Current Phenomena.'

If such is the case it can only be explained by the fact that the theory of the simple alternator has not been completely worked out.

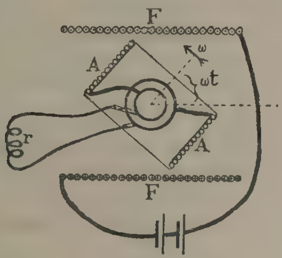
In the following paper this is done for an alternator with a uniform field by means of a new application of the vector method in which all the harmonics of a periodic function are dealt with simultaneously.

In the same is shown how to take account of hysteresis and eddy currents, and the theory of the action of dampers in reducing the heating in the field is also given.

The theory of the alternate current synchronous motor is also dealt with.

1. Let two coils be arranged as indicated in fig. 1, one of them, F, called the field coil, being fixed and having a battery of constant E.M.F. $= \eta$ in its circuit, the other, A, called the

Fig. 1.



armature coil, fitted in the usual way with slip rings for connexion to an external circuit and being rotated by power at a constant angular velocity ω round a fixed axis which is perpendicular to its own axis of figure and to the direction of the lines of force of F, and which passes through its own centre. It is required to determine completely the currents that flow in both A and F.

Let x and ξ be the currents at any instant in A and F respectively, and let the mutual inductance of A and F when their axes are coincident be m, and hence $m \cos \omega t$ at the time t. Also let r and l be the total resistance and self-inductance of the A circuit, and ρ, λ similar quantities for the F circuit.

Then, when the armature is being driven at constant

angular velocity ω, and x and ξ are flowing, the total number of lines linked on A is

$$lx + m\xi \cos \omega t,$$

and the number linked on F is

$$\lambda\xi + mx \cos \omega t.$$

Hence

$$\left.\begin{aligned} rx + \frac{d}{dt}\{lx + m\xi \cos \omega t\} &= 0 \\ \rho\xi + \frac{d}{dt}\{\lambda\xi + mx \cos \omega t\} &= \eta \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(I.)}$$

where η is the applied steady E.M.F. in the F circuit.

2. If we assume as the solution of these equations

$$x = \frac{x_0}{2} + x_1 \sin(\omega t + c_1) + x_2 \sin(2\omega t + c_2) + x_3 \sin(3\omega t + c_3) + \&c.,$$

$$\xi = \frac{\xi_0}{2} + \xi_1 \sin(\omega t + \gamma_1) + \xi_2 \sin(2\omega t + \gamma_2) + \xi_3 \sin(3\omega t + \gamma_3) + \&c.,$$

we can see at once on substitution that $\rho\xi_0 = 2\eta$, and that $x_0 = 0$, and it will be shown afterwards, § 15, that when $x_0 = 0$ then $\xi_1, x_2, \xi_3, x_4, \xi_5$, &c., vanish, or in words, when $x_0 = 0$ only odd harmonics appear in x, and only even ones in ξ. Let us therefore take

$$\left.\begin{aligned} x &= x_1 \sin(\omega t + c_1) + x_3 \sin(3\omega t + c_3) + x_5 \sin(5\omega t + c_5) + \&c. \\ \xi &= \frac{\xi_0}{2} + \xi_2 \sin(2\omega t + \gamma_2) + \xi_4 \sin(4\omega t + \gamma_4) + \&c. \end{aligned}\right\} \text{(II.)}$$

Now any harmonic in either x or ξ, for instance $x_q \sin(q\omega t + c_q)$, being completely specified by x_q, c_q, and q, can, when its order q is known, be represented by the vector drawn from the origin in any reference plane to the point in that plane whose polar coordinates are x_q, c_q, twice the constant term in ξ being represented in the same plane by the vector to the point $\xi_0, \frac{\pi}{2}$

The form of solution (II.) assumed may now be written

$$\left.\begin{aligned} x &= \mathbf{a}_1 + \mathbf{a}_3 + \mathbf{a}_5 + \&c. \\ \xi &= \frac{\alpha_0}{2} + \alpha_2 + \alpha_4 + \alpha_6 + \&c. \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(III.)}$$

where $\mathbf{a}_1$, $\mathbf{a}_3$, &c., α_0, α_2, α_4, &c., are vectors whose orders are indicated by the subscribed numbers. Of these, one only, namely α_0, is known, as it is drawn to the point whose polar co-ordinates are ξ_0, $\frac{\pi}{2}$, where $\xi_0 = \frac{2\eta}{\rho}$. The others have to be determined.

Note a.—In the sequel it will sometimes happen that a vector, say a_q, originally assumed of order q, will be used to represent an harmonic of a different order, say $q+1$. In such a case it will be written $(\mathbf{a}_q)_{q+1}$; thus

$$\mathbf{a}_q = x_q \sin (q\omega t + c_q),$$

but

$$(\mathbf{a}_q)_{q+1} = x_q \sin \{(q+1)\omega t + c_q\}.$$

Note b.—The length of a vector α will be written as $\bar{\alpha}$ (*i. e.* with the bar); thus $\bar{\mathbf{a}}_3 = x_3$, unless in cases where no ambiguity can arise, when α simply will be written for the length of the vector α.

3. If we agree to indicate by ι^θ the operation of rotating any vector to which it is prefixed through an angle θ in the positive direction, then

$$\iota^\pi \alpha = -\alpha \quad \text{or} \quad \iota^\pi = -1,$$

and

$$\iota^\theta \alpha = (\cos\theta + \iota^{\frac{\pi}{2}} \sin\theta)\alpha, \quad \text{or} \quad \iota^\theta = \cos\theta + \iota^{\frac{\pi}{2}} \sin\theta.$$

Also, if $t = \mathrm{D}\iota^f$, $t\alpha$ is the vector obtained by increasing α in length D times and then rotating the increased vector through an angle f in the positive direction.

Plane vector operators such as t are well known to be subject to the same rules as ordinary algebraical symbols.

Again, the sum of two operators $a_1\iota^{\theta_1}$, $a_2\iota^{\theta_2}$, can be expressed as a single operator $\mathrm{A}\iota^\psi$, say, that is

$$\mathrm{A}\iota^\psi \alpha = a_1\iota^{\theta_1}\alpha + a_2\iota^{\theta_2}\alpha,$$

where α is any vector.

Using the expression for ι^θ given above,

$$\mathrm{A}(\cos\psi + \iota^{\frac{\pi}{2}} \sin\psi)\alpha$$
$$= a_1(\cos\theta_1 + \iota^{\frac{\pi}{2}} \sin\theta_1)\alpha + a_2(\cos\theta_2 + \iota^{\frac{\pi}{2}} \sin\theta_2)\alpha,$$

so that

$$\text{A} \cos \psi = a_1 \cos \theta_1 + a_2 \cos \theta_2 ;$$
$$\text{A} \sin \psi = a_1 \sin \theta_1 + a_2 \sin \theta_2.$$

Hence

$$\text{A}^2 = a_1{}^2 + a_2{}^2 + 2a_1a_2 \cos (\theta_1 - \theta_2),$$

and

$$\tan \psi = \frac{a_1 \sin \theta_1 + a_2 \sin \theta_2}{a_1 \cos \theta_1 + a_2 \cos \theta_2}.$$

Again, if

$$\alpha_p = \xi_p \sin (p\omega t + \gamma_p),$$

then

$$\frac{d}{dt}(\alpha_p) = p\omega\iota^{\frac{\pi}{2}}\alpha_p,$$

seeing that

$$\frac{d}{dt}(\alpha_p) = p\omega\xi_p \sin \left(p\omega t + \gamma_p + \frac{\pi}{2}\right).$$

Hence for x and ξ as expressed in § 2

$$\frac{dx}{dt} = \omega\iota^{\frac{\pi}{2}}\Sigma q\mathbf{a}_q, \quad \frac{d\xi}{dt} = \omega\iota^{\frac{\pi}{2}}\Sigma p\alpha_p.$$

4. By means of the formula

$$2 \sin a \cos b = \sin (a+b) + \sin (a-b)$$

it is easy to show that $2x \cos \omega t$, where x is the **a** series of odd order vectors in § 2, is represented by the series of even order vectors of which the one of the pth order is the vector sum of $\mathbf{a}_{p-1}$ and $\mathbf{a}_{p+1}$, or that

$$2x \cos \omega t = (\mathbf{a}_1)_0 + (\mathbf{a}_1 + \mathbf{a}_3)_2 + (\mathbf{a}_3 + \mathbf{a}_5)_4 + (\mathbf{a}_5 + \mathbf{a}_7)_6 + \&\text{c.},$$

$(\mathbf{a}_1)_0$ being the resolved part of $\mathbf{a}_1$ along the y axis, that is along the direction of vectors of zero order (see Note a, § 2).

Similarly

$$2\xi \cos \omega t = (\alpha_0 + \alpha_2)_1 + (\alpha_2 + \alpha_4)_3 + (\alpha_4 + \alpha_6)_5 + \&\text{c.}$$

Again, by means of the formula

$$2 \sin a \sin b = \cos (a-b) - \cos (a+b),$$

it is easy to show that

$$2\xi \sin \omega t = \iota^{-\frac{\pi}{2}}(\alpha_0 - \alpha_2)_1 + \iota^{-\frac{\pi}{2}}(\alpha_2 - \alpha_4)_3 + \iota^{-\frac{\pi}{2}}(\alpha_4 - \alpha_6)_5 + \&\text{c.}$$
$$= \iota^{-\frac{\pi}{2}}\Sigma(\alpha_{q-1} - \alpha_{q+1})_q,$$

where q is odd with a similar result for the product $2x \sin \omega t$.

5. If we now substitute the vector expressions from §§ 2, 3, 4, in equations I. and equate separately to zero each set of vector terms of the same order we obtain the two series of vector equations

$$r\mathrm{a}_q + q\omega\iota^{\frac{\pi}{2}}\left\{l\mathrm{a}_q + \frac{m}{2}(\alpha_{q-1} + \alpha_{q+1})\right\} = 0, \qquad \text{(IV.)}$$

$$\rho\alpha_p + p\omega\iota^{\frac{\pi}{2}}\left\{\lambda\alpha_p + \frac{m}{2}(\mathrm{a}_{p-1} + \mathrm{a}_{p+1})\right\} = 0, \qquad \text{(V.)}$$

together with $\xi_0 = 2\eta/\rho$, where q is any odd number and p any even number.

From IV. we deduce a series of equations of the type

$$\alpha_{q-1} + 2\frac{q\omega l - r\iota^{\frac{\pi}{2}}}{q\omega m}\mathrm{a}_q + \alpha_{q+1} = 0,$$

or

$$\alpha_{q-1} + t_q\,\mathrm{a}_q + \alpha_{q+1} = 0,$$

where t_q is the operator $\mathrm{D}_q\iota^{-f_q}$, in which

$$\mathrm{D}_q \cos f_q = 2\frac{l}{m}, \quad \mathrm{D}_q \sin f_q = \frac{2r}{q\omega m},$$

that is

$$\mathrm{D}^2 = \frac{4}{m^2}\left(l^2 + \frac{r^2}{q^2\omega^2}\right), \quad \tan f_q = \frac{r}{q\omega l}.$$

Similarly from V. we deduce the series

$$\mathrm{a}_{p-1} + \tau_p\,\alpha_p + \mathrm{a}_{p+1} = 0$$

where τ_p is the operator $\Delta_p\,\iota^{-\phi_p}$, in which

$$\Delta_p \cos\phi_p = 2\frac{\lambda}{m}, \quad \Delta_p \sin\phi_p = \frac{2\rho}{p\omega m}$$

or

$$\Delta_p^2 = \frac{4}{m^2}\left(\lambda^2 + \frac{\rho^2}{p^2\omega^2}\right), \quad \tan\phi_p = \frac{\rho}{p\omega\lambda}.$$

Note that the vector equations in this paragraph are equations connecting the different vectors, considered purely as vectors, without any reference to the order of the harmonic they originally represented.

6. We have thus obtained the following infinite series of equations connecting the vectors used to represent x and ξ :—

$$\left.\begin{aligned} t_1\mathbf{a}_1 + \alpha_2 \qquad\qquad\qquad\qquad &= -\alpha_0 \\ \mathbf{a}_1 + \tau_2\alpha_2 + \mathbf{a}_3 \qquad\qquad\qquad &= 0 \\ \alpha_2 + t_3\mathbf{a}_3 + \alpha_4 \qquad\qquad &= 0 \\ \mathbf{a}_3 + \tau_4\alpha_4 + \mathbf{a}_5 \qquad &= 0 \\ \alpha_4 + t_5\mathbf{a}_5 + \alpha_6 &= 0 \\ \&c., \&c. \end{aligned}\right\} \quad \text{(VI.)}$$

And as it is well known that algebraic methods are applicable to plane vector operators of the type here made use of, we obtain the following infinite determinant vector solution for $\mathbf{a}_1$, namely,

$$\begin{vmatrix} t_1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & \tau_2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & t_3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & \tau_4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & t_5 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 1 & \tau_6 & \ldots \\ & & \&c., \&c. \end{vmatrix} \mathbf{a}_1 = - \begin{vmatrix} \tau_2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & t_3 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & \tau_4 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & t_5 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \tau_6 & 1 & \ldots \\ & & \&c., \&c. \end{vmatrix} \alpha_0$$

or

$$P_1\mathbf{a}_1 = -\Pi_2\alpha_0$$

where P_1 is the infinite determinant operator whose leading term is t_1, and Π_2 that whose leading term is τ_2.

7. P_1, Π_2, P_3, Π_4, &c., being the determinants whose leading terms are t_1, τ_2, t_3, τ_4, &c. respectively, we find at once by expanding that

$$P_1 = t_1\,\Pi_2 - P_3$$

$$\Pi_2 = \tau_2\,P_3 - \Pi_4, \ \&c.,$$

hence

$$\frac{P_1}{\Pi_2} = t_1 - \frac{P_3}{\Pi_2} = t_1 - \frac{1}{\Pi_2/P_3}$$

$$= t_1 - \cfrac{1}{\tau_2 - \cfrac{1}{t_3 - \cfrac{1}{\tau_4 - \&c.}}}$$

$$= S_1 \text{ (say)},$$

so that

$$\mathbf{a}_1 = -\frac{1}{S_1}\alpha_0,$$

where S_1 is the infinite continued fraction operator whose leading term is t_1.

Again, if

$$\Sigma_2 = \tau_2 - \cfrac{1}{t_3 - \cfrac{1}{\tau_4 - \cfrac{1}{t_5 - \&c.}}}$$

$$S_3 = t_3 - \cfrac{1}{\tau_4 - \cfrac{1}{t_5 - \cfrac{1}{\tau_6 - \&c.}}}$$

$$\Sigma_4 = \tau_4 - \cfrac{1}{t_5 - \&c.},$$

and so on, we find in a similar way, or by making use of equations VI., that

$$\alpha_2 = -\frac{1}{\Sigma_2}\mathbf{a}_1$$

$$\mathbf{a}_3 = -\frac{1}{S_3}\alpha_2 \text{ \&c., \&c.}$$

Hence for the complete solution we have

$$\alpha_0 = -S_1\mathbf{a}_1 = S_1\Sigma_2\alpha_2 = -S_1\Sigma_2 S_3\mathbf{a}_3 = \&c.,$$

which gives $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$, &c., α_2, α_4, α_6, &c., in terms of the known vector α_0 provided the continued fraction operators S_1, Σ_2, S_3, &c., are determinate.

It is easily seen that they are determinate for when q becomes a large number

$$t_q = 2\frac{l}{m} = t_{q+2}$$

$$\tau_{q+1} = 2\frac{\lambda}{m} = \tau_{q+3} \text{ (see § 5)};$$

that is, S_1, Σ_2, S_3, Σ_4, &c. are recurring continued fraction operators, and the recurring elements when reached are simple numbers.

8. The form of solution obtained is one very easy of practical application. In computing the continued fraction operators just so many of their known t, τ elements (see § 5) need be taken account of as are necessary to give the required degree of approximation.

Moreover, in the case of a practical alternator, as the resistance of the field-magnet coils is negligible relative to

their reactance, all the τ elements are practically simple numbers independent of the field resistance, while for the t elements q is never a large number when t_{q+2} differs little from t_q. So that we could obtain S_q with considerable accuracy by assuming the recurring stage to be reached, and therefore

$$S_q = t_q - \frac{1}{\tau_{q+1} - \frac{1}{S_q}},$$

which gives the quadratic in S_q

$$S_q^2 - t_q S_q + \frac{t_q}{\tau_{q+1}} = 0,$$

from which S_q can be obtained by ordinary algebra.

[In solving this quadratic the two operators that come under the square-root symbol will have to be reduced by the addition theorem in § 3 to a single operator, $a\iota^{\mathrm{b}}$ say, and the square root of this is $\sqrt{a}\iota^{\frac{1}{2}\mathrm{b}}$.]

If S_q obtained in either of these ways be $= s_q\iota^{-\mathrm{b}_q}$, then as

$$\Sigma_{q-1} = \tau_{q-1} - \frac{1}{S_q} = \tau_{q-1} - \frac{1}{s_q}\iota^{\mathrm{b}_q},$$

Σ_{q-1} can be obtained by the addition theorem, and so on for S_{q-2}, Σ_{q-3}, &c., up to S_1. Let the results be written

$$S_1 = s_1\iota^{-\mathrm{b}_1}, \quad \Sigma_2 = \sigma_2\iota^{-\beta_2}, \quad S_3 = s_3\iota^{-\mathrm{b}_3}, \text{ \&c.,}$$

and in general

$$S_q = s_q\iota^{-\mathrm{b}_q}, \quad \Sigma_p = \sigma_p\iota^{-\beta_p}.$$

9. As α_0 is the vector of length $2\eta/\rho$ lying along the axis of y (phase $=\pi/2$), and as

$$\mathfrak{a}_1 = -\frac{1}{S_1}(\alpha_0)_1 = -\frac{1}{s_1}\iota^{\mathrm{b}_1}(\alpha_0)_1$$

$$\mathfrak{a}_2 = -\frac{2\eta}{s_1\rho}\sin\left(\omega t + \frac{\pi}{2} + \mathrm{b}_1\right) = \frac{2\eta}{s_1\rho}\sin\left(\omega t - \frac{\pi}{2} + \mathrm{b}_1\right).$$

Again, as

$$\alpha_2 = -\frac{1}{\Sigma_2}(\mathfrak{a}_1)_2 = -\frac{1}{\sigma_2}\iota^{\beta_2}(\mathfrak{a}_1)_2,$$

$$\alpha_2 = \frac{2\eta}{s_1\sigma_2\rho}\sin\left(2\omega t + \frac{\pi}{2} + \mathrm{b}_1 + \beta_2\right);$$

similarly

$$\mathfrak{a}_3 = \frac{2\eta}{s_1\sigma_2 s_3\rho}\sin\left(3\omega t - \frac{\pi}{2} + \mathfrak{b}_1 + \beta_2 + \mathfrak{b}_3\right);$$

$$\alpha_4 = \frac{2\eta}{s_1\sigma_2 s_3\sigma_4\rho}\sin\left(4\omega t + \frac{\pi}{2} + \mathfrak{b}_1 + \beta_2 + \mathfrak{b}_3 + \beta_4\right);$$

$$\mathfrak{a}_5 = \frac{2\eta}{s_1\sigma_2 s_3\sigma_4 s_5\rho}\sin\left(5\omega t - \frac{\pi}{2} + \mathfrak{b}_1 + \beta_2 + \mathfrak{b}_3 + \beta_4 + \mathfrak{b}_5\right);$$

&c., &c.

And substituting these values in

$$x = \mathfrak{a}_1 + \mathfrak{a}_3 + \mathfrak{a}_5 + \&c., \qquad \xi = \frac{\eta}{\rho} + \alpha_2 + \alpha_4 + \&c.,$$

we obtain the armature and field currents in the usual trigonometrical form of expression.

It is worth while drawing attention to the fact that the period of the alternating current induced in the field circuit is half that of the armature current, and that it contains all harmonics, both odd and even, relative to its own fundamental, and so its wave-form will in general be unsymmetrical with respect to the time-axis.

10. The total E.M.F. E generated in the armature circuit being equal to $-\frac{d}{dt}(m\xi\cos\omega t)$,

$$\mathrm{E} = -\frac{m}{2}\frac{d}{dt}\Sigma(\alpha_{q-1} + \alpha_{q+1})_q, \quad \text{(see § 4)}$$

$$= -\frac{\omega m}{2}\iota^{\frac{\pi}{2}}\Sigma q(\alpha_{q-1} + \alpha_{q+1})_q.$$

Also as $\qquad \alpha_{q-1} + t_q\mathfrak{a}_q + \alpha_{q+1} = 0$

$$\mathrm{E} = \frac{\omega m}{2}\iota^{\frac{\pi}{2}}\Sigma q t_q \mathfrak{a}_q.$$

And in either of these formulæ the trigonometrical expressions in § 9 for the vectors can be substituted.

In the first, however, it must be noted that both α_{q-1} and α_{q+1} are to be taken of order q (odd). Thus the fundamental harmonic of E is

$$= \frac{\omega m\eta}{\rho}\left\{\sin\omega t + \frac{1}{s_1\sigma_2}\sin(\omega t + \mathfrak{b}_1 + \beta_2)\right\}$$

from the first expression, or

$$= \frac{\omega m \eta}{\rho} \frac{D_1}{s_1} \sin(\omega t + \mathfrak{b}_1 - f_1)$$

from the second, as $t_1 = D_1 \iota^{-f_1}$.

Similarly the total alternating E.M.F. (H say) generated in the field circuit is given by either

$$H = -\frac{\omega m}{2} \iota^{\frac{\pi}{2}} \Sigma p(\mathfrak{a}_{p-1} + \mathfrak{a}_{p+1})_p$$

or $$H = \frac{\omega m}{2} \iota^{\frac{\pi}{2}} \Sigma p \tau_p \alpha_p,$$

so that the fundamental harmonic of H is equal to either

$$\frac{2\omega m\eta}{\rho}\left[\frac{1}{s_1} \sin(2\omega t + \mathfrak{b}_1 + \pi) + \frac{1}{s_1 \sigma_2 s_3} \sin(2\omega t + \mathfrak{b}_1 + \beta_2 + \mathfrak{b}_3 + \pi)\right]$$

or $$\frac{2\omega m\eta}{\rho} \frac{\Delta_2}{s_1 \sigma_2} \sin(2\omega t + \mathfrak{b}_1 + \beta_2 - \phi_2 + \pi).$$

11. The mean value of the product

$$\sin(a\omega t + \theta) \sin(b\omega t + \phi)$$

being zero when a and b are unequal and $\frac{1}{2}\cos(\theta - \phi)$ when a and b are equal, we find that the mean value of x^2 where $x = \Sigma \mathfrak{a}_q$ is

$$= \tfrac{1}{2} \Sigma \bar{\mathfrak{a}}_q^2 ;$$

and the mean value of ξ^2, where $\xi = \frac{\alpha}{2} + \Sigma \alpha_p$, is

$$= \frac{\bar{\alpha}_0^2}{4} + \tfrac{1}{2} \Sigma \bar{\alpha}_p^2.$$

Again, for the same reason, if α and β be any two vectors *representing harmonics of the same order* and if $S\alpha\beta$ be the product of the lengths of α and β into the sine of the angle from α to β *measured in the positive direction*, then the mean value of the product

$$\iota^{\frac{\pi}{2}} \alpha \text{ into } \beta$$

is $$= \tfrac{1}{2} S\alpha\beta = -\tfrac{1}{2} S\beta\alpha.$$

Applying these principles to the determination of the

mean value $\overline{Ex}$ of the product of E and x, that is of the electrical power developed in the armature circuit, we find from the first expression for E in § 10 that

$$\overline{Ex} = -\frac{\omega m}{4}\Sigma q \mathrm{S}(\alpha_{q-1} + \alpha_{q+1})\mathrm{a}_q$$

$$= -\frac{\omega m}{4}\{\mathrm{S}\alpha_0\mathrm{a}_1 - \mathrm{S}\mathrm{a}_1\alpha_2 + 3\mathrm{S}\alpha_2\mathrm{a}_3 - 3\mathrm{S}\mathrm{a}_3\alpha_4 + 5\mathrm{S}\alpha_4\mathrm{a}_5 - 5\mathrm{S}\mathrm{a}_5\alpha_6 + \&c.\},$$

and from the second expression for E that

$$\overline{Ex} = \frac{\omega m}{4}\Sigma q\mathrm{S}(t_q\mathrm{a}_q \,.\, \mathrm{a}_q\,.) = \frac{\omega m}{4}\Sigma q\mathrm{D}_q\mathrm{S}(\iota^{-f_q}\mathrm{a}_q \,.\, \mathrm{a}_q\,.)$$

$$= \frac{\omega m}{4}\Sigma\bar{\mathrm{a}}_q^2 q\mathrm{D}_q \sin f_q = \tfrac{1}{2}r\Sigma\bar{\mathrm{a}}_q$$

as $\mathrm{D}_q \sin f_q = \dfrac{2r}{q\omega m}$ (see § 5)

= the heat developed in the armature circuit.

Similarly the total electrical power $\overline{(\mathrm{H}+\eta)\xi}$ developed in the field circuit is given by

$$\overline{(\mathrm{H}+\eta)\xi} = \frac{\eta\alpha_0}{2} - \frac{\omega m}{4}\Sigma p\mathrm{S}(\mathrm{a}_{p-1} + \mathrm{a}_{p+1})\alpha_p$$

$$= \rho\frac{\bar{\alpha}_0^2}{4} - \frac{\omega m}{4}\{2\mathrm{S}\mathrm{a}_1\alpha_2 - 2\mathrm{S}\alpha_2\mathrm{a}_3 + 4\mathrm{S}\mathrm{a}_3\alpha_4 - 4\mathrm{S}\alpha_4 a_5 + \&c.\}$$

or by

$$\overline{(\mathrm{H}+\eta)\xi} = \frac{\eta\alpha_0}{2} + \frac{\omega m}{4}\Sigma p\mathrm{S}(\tau_p\alpha_p \,.\, \alpha_p\,.)$$

which reduces to

$$\overline{(\mathrm{H}+\eta)\xi} = \rho\frac{\bar{\alpha}_0^2}{4} + \frac{\rho}{2}(\bar{\alpha}_2^2 + \bar{\alpha}_4^2 + \bar{\alpha}_6^2 + \&c.)$$

= total heat developed in the field circuit

and is made up of two parts, the first $= \rho\dfrac{\alpha_0^2}{4}$ due to the direct exciting current and the second $= \dfrac{\rho}{2}(\bar{\alpha}_2^2 + \bar{\alpha}_4^2 + \bar{\alpha}_6^2 + \&c.)$ due to the induced alternating field-current.

Adding the first expressions for $\overline{Ex}$ and $\overline{(\mathrm{H}+\eta)\xi}$ and cancelling $\eta\bar{\xi}$ against $\rho\dfrac{\bar{\alpha}_0^2}{4}$ we find that

$$\overline{Ex} + \overline{\mathrm{H}\xi} = -\frac{\omega m}{4}\{\mathrm{S}\alpha_0\mathrm{a}_1 + \mathrm{S}\mathrm{a}_1\alpha_2 + \mathrm{S}\alpha_2\mathrm{a}_3 + \mathrm{S}\mathrm{a}_3\alpha_4 + \&c.\}.$$

12. The torque exerted at any instant in driving the alternator is

$$= -x\xi\frac{d}{d(\omega t)}\{m\cos\omega t\} = mx\xi\sin\omega t$$

$$= \frac{m}{2}\Sigma\mathbf{a}_q \times \iota^{-\frac{\pi}{2}}\Sigma(\alpha_{q-1}-\alpha_{q+1})_q \qquad \text{(see § 4)}$$

$$= -\frac{m}{2}\Sigma\mathbf{a}_q \times \iota^{\frac{\pi}{2}}\Sigma(\alpha_{q-1}-\alpha_{q+1})_q.$$

Taking the mean value of this product we find that the mean driving torque T is given by

$$\mathrm{T} = -\frac{m}{4}\{\mathrm{S}\alpha_0\mathbf{a}_1 + \mathrm{S}\mathbf{a}_1\alpha_2 + \mathrm{S}\alpha_2\mathbf{a}_3 + \mathrm{S}\mathbf{a}_3\alpha_4 + \&\text{c.}\},$$

which result, combined with the last one obtained in § 11, gives the power equation

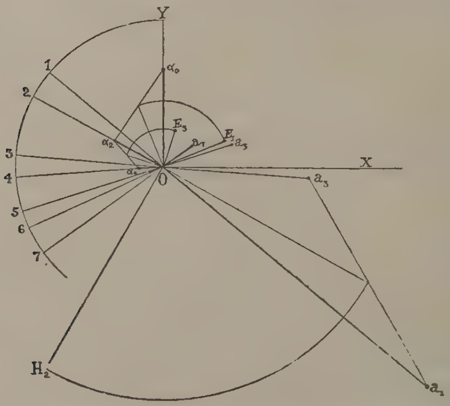
$$\omega\mathrm{T} = \overline{\mathrm{E}x} + \overline{\mathrm{H}\xi}$$

as it ought.

13. The solution obtained can be represented geometrically in an interesting way as follows:—

Take two lines OX, OY, fig. 2, at right angles. Measure

Fig. 2.



off from OY in the positive direction the angles $\mathrm{YO1} = \mathbf{b}_1$, $102 = \beta_2$, $203 = \mathbf{b}_3$, $304 = \beta_4$, &c., where $\mathbf{b}_1$, β_2, $\mathbf{b}_3$, β_4, &c., are the angles determined in § 8.

In OY take $O\alpha_0 = 2\eta/\rho = 2 \times$ exciting current. Produce 10 through O to $\mathbf{a}_1$ so that $O\mathbf{a}_1 = \frac{O\alpha_0}{s_1}$. In O2 take $O\alpha_2 = O\mathbf{a}_1/\sigma_2$. Produce 3O through O to $\mathbf{a}_3$ so that $O\mathbf{a}_3 = \frac{O\alpha_2}{s_3}$, and so on where s_1, σ_2, s_3, σ_4, &c., are the quantities determined in § 8.

Then the vectors to $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$, &c., represent completely in amplitude and phase the different harmonics of the armature current, the subscribed numbers indicating the orders of the harmonics; and those to α_2, α_4, α_6, &c., represent completely in the same way the different harmonics of the induced alternating field current.

Again (see § 10), if we rotate the vector drawn to the middle point of $\alpha_0\alpha_2$ backwards through a right angle, we obtain the vector OE_1 that represents $\frac{1}{m\omega}$ into the first harmonic of the total E.M.F. E generated in the armature; and if we rotate backwards through $\pi/2$ the vector to the middle point of $\alpha_2\alpha_4$ we obtain the vector OE_3 that represents $\frac{1}{3\omega m}$ into the third harmonic of E; and similarly for the other harmonics of E.

In the same way, by rotating backwards through $\pi/2$ the vector to the middle point of $\mathbf{a}_1\mathbf{a}_3$, we obtain the vector OH_2 that represents $\frac{1}{2\omega m}$ into the fundamental harmonic of the E.M.F. H induced in the field circuit; and so on for the other harmonics of H.

Again, as $S\alpha\beta$ is = twice the area of the triangle whose sides are α and β and is positive if β follows α in rotation order in the diagram, the mean torque exerted on the generator by the driver (see § 12) is equal to $\frac{1}{2}m$ into the sum of the areas of the triangles $\alpha_0 O\mathbf{a}_1$, $\mathbf{a}_1 O\alpha_2$, $\alpha_2 O\mathbf{a}_3$, $\mathbf{a}_3 O\alpha_4$, &c., these triangles, in the case of any generator, being all taken as positive.

14. When, for any generator, the t, τ operators have been calculated for a particular load (see § 5), a geometrical solution can easily be obtained to a high degree of accuracy by aid of a ruler, scale, slide-rule, and protractor.

Thus if we neglect the harmonics α_8, $\mathbf{a}_9$, α_{10}, &c., then, drawing any vector from the origin to represent $\mathbf{a}_7$ we can construct for α_6 as $\alpha_6 = -t_7\mathbf{a}_7$ (see § 6). From α_6 we can construct for $\tau_6\alpha_6$ and as $\mathbf{a}_5 + \tau_6\alpha_6 + \mathbf{a}_7 = 0$ the triangle of vectors gives us $\mathbf{a}_5$. Proceeding in this way, we obtain in succession α_4, $\mathbf{a}_3$, α_2, $\mathbf{a}_1$, α_0, which represent the harmonics of x and ξ correctly as regards relative phase and relative amplitude. But as α_0 must be equal to twice the exciting current we have a scale for our diagram, and hence obtain a complete solution.

The fact that for a practical alternator the τ operators may be taken as pure numbers (see §§ 8, 22), renders this method of solution both easy and expeditious.

15. If a source of constant E.M.F. be included in the armature circuit as well as in the field circuit of the simple alternator indicated in fig. 1, equations I. § 1 become

$$rx + \frac{d}{dt}(lx + m\xi \cos \omega t) = e$$

$$\rho\xi + \frac{d}{dt}(\lambda\xi + mx \cos \omega t) = \eta,$$

and both the armature and field currents will now contain harmonics of all orders, odd and even. In this case we assume that

$$x = \frac{\mathbf{a}_0}{2} + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \&c.$$

$$\xi = \frac{\alpha_0}{2} + \alpha_1 + \alpha_2 + \alpha_3 + \&c.,$$

where $\mathbf{a}_0$ is the vector to the point whose polar coordinates are $2e/r$, $\pi/2$, and α_0 is, as before, the vector to the point $2\eta/\rho$, $\pi/2$. The other vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, &c., α_1, α_2, α_3, &c., have to be determined.

On substituting for x and ξ in the above equations it will be found that the odd order vectors in x and the even order ones in ξ are determined by the same equations (IV. § 6) as when $e=0$, and are completely independent of the even order vectors in x and the odd order ones in ξ, these latter depending only on e and vanishing with e.

This being so, $\mathbf{a}_1$, α_2, $\mathbf{a}_3$, α_4, &c., are given by the solution already obtained, and α_1, $\mathbf{a}_2$, α_3, $\mathbf{a}_4$, &c., will be given by a

similar solution the equations expressing which may be written down from symmetry.

Thus the complete solution is given by

$$\alpha_0 = -S_1\mathbf{a}_1 = S_1\Sigma_2\alpha_2 = -S_1\Sigma_2S_3\mathbf{a}_3 = \&c.,$$
$$\mathbf{a}_0 = -\Sigma_1\alpha_1 = \Sigma_1S_2\mathbf{a}_2 = -\Sigma_1S_2\Sigma_3\alpha_3 = \&c.,$$

where α_0 is the vector to $2\eta/\rho$, $\pi/2$, as before,

and $\mathbf{a}_0$ is the vector to $2e/r$, $\pi/2$.

Note.—In the former case ($e=0$) the t operators were all of odd and the τ ones of even orders. In this case the operators of either class are of both orders.

The translation from the above vector solution to the ordinary sine form follows as in § 10.

16. In the preceding solutions the magnetic fluxes have been assumed to be in phase with the magnetizing current-turns, and so iron loss due to hysteresis and eddy currents has been neglected. To take account of the latter the interpretation of the well-known relation $B=\mu H$ connecting steady magnetizing force and induction produced has to be modified.

The induction produced by $H=H_1 \sin(\omega t + c_1)$ is known to be of the form

$$B = m_1H_1 \sin(\omega t + c_1 - \delta_1) + \text{higher harmonics},$$

and attending only to the fundamental harmonic in B, if H be represented as explained in § 2 by the vector h_1, and B by the vector b_1, then the above trigonometrical relation may be written

$$b_1 = \mu_1 h_1$$

where μ_1 is the operator $m_1\iota^{-\delta_1}$.

In a former paper * by me was shown how these *permeability operators*, as they may be called, can be determined. They depend on the character of the iron and the thickness of the laminæ, on the amplitude and period of the fundamental harmonic of the induction oscillation they refer to, and to some extent on the wave form of the latter.

For the purposes of the following discussion we will

* "Variation of Magnetic Hysteresis with Frequency," Phil. Mag. Jan. 1905.

assume that when the magnetizing force

$$H = h_1 + h_3 + h_5 + \&c.$$

produces the induction

$$B = b_1 + b_3 + b_5 + \&c.,$$

then $b_1 = \mu_1 h_1$, $b_3 = \mu_3 h_3$, $b_5 = \mu_5 h_5$, &c., where the μ's are operators of the type given by

$$\mu_q = m_q \iota^{-\delta_q}.$$

[This assumption as regards all the harmonics of B but the fundamental is not strictly in accordance with what is known concerning the behaviour of laminated iron under periodic magnetizing forces, for b_3, b_5, &c., depend, at any rate for high values of b_1, more on b_1 than on h_3, h_5, &c. At the same time it is hoped that the following discussion may be of some value.]

In general if $H = \Sigma h_q$ produce $B = \Sigma b_q$, as the total iron loss per c.c. per cycle due to both hysteresis and eddy currents is

$$\frac{1}{4\pi}\int H dB = \frac{1}{4\pi}\int_0^T H \frac{dB}{dt} dt,$$

where T is the period, the total iron loss per c.c. per second is

$$= \frac{1}{4\pi}. \quad \text{Average value of product } H\frac{dB}{dt}$$

$$= \frac{1}{4\pi}. \quad \text{Av. product } \Sigma h_q \text{ into } \iota^{\frac{\pi}{2}} \Sigma q \omega b_q$$

$$= \frac{\omega}{8\pi}\{Sb_1h_1 + 3Sb_3h_3 + 5Sb_5h_5 + \&c.\}$$

(See § 11)

$$= \frac{\omega}{8\pi}\Sigma q \bar{b}_q \bar{h}_q \sin\delta_q.$$

Again, it is well known that if the steady magnetizing current-turns nx act on a magnetic circuit composed of different materials, the flux F produced is given by

$$F = \frac{4\pi nx}{\Sigma \dfrac{L}{A\mu}},$$

where the L, A's are the lengths and sectional areas of the

different portions of the circuit, and the μ's are the permeabilities of these portions for the particular flux densities in them.

If now the magnetizing current be an alternating one, that is if $x = x_1 \sin(\omega t + c_1) = \mathbf{a}_1$ (a vector) the same equation will give the corresponding harmonic of the flux produced, but the μ's are now the permeability operators, for the different portions of the circuit.

$\Sigma \frac{L}{A\mu}$ can, by the addition theorem in § 3, be reduced to a angle operator so that if the flux f_1 (vector) be produced in any magnetic circuit by the current-turns $n\mathbf{a}_1$ we have always a relation of the form

$$f_1 = G_1 n\mathbf{a}_1$$

where G_1 is an operator of the form $g_1\iota^{-\delta}$ which can be determined.

Hence, following the assumption already made, if the magnetizing current-turns

$$nx = n(\mathbf{a}_1 + \mathbf{a}_3 + \mathbf{a}_5 + \&c.)$$

produce in a magnetic circuit the flux

$$F = f_1 + f_3 + f_5 + \&c.$$

then $$f_1 = nG_1\mathbf{a}_1,\ f_3 = nG_3\mathbf{a}_3,\ \&c.$$

where the G's are operators of the type given by $G_q = g_q\iota^{-\delta_q}$.

In the above the back E.M.F., e say, in the magnetizing coils due to variation of flux is

$$e = n\frac{dF}{dt} = n\omega\iota^{\frac{\pi}{2}}\Sigma q f_q$$

and the power absorbed, that is the total iron loss per sec. in the magnetic circuit, is the mean value of ex, that is of the product

$$\Sigma\mathbf{a}_q \text{ into } n\omega\iota^{\frac{\pi}{2}}\Sigma q f_q$$

which (see § 11) $$= \tfrac{1}{2}n\omega\Sigma(q\mathrm{S}f_q \,.\, \mathbf{a}_q \,.)$$

$$= \tfrac{1}{2}\omega\Sigma\left(q\frac{f_q^2}{g_q}\sin\delta_q\right).$$

17. As an example let us determine the G operator for a magnetic circuit of uniform cross section = 100 cm.², made

up of 40 cm. length of laminated iron and two air-gaps each 1 mm. when B maximum = 5000 and the frequency 30.

In the paper already quoted we find for a sample of No. 26 iron well insulated between the laminæ when B_{max} = 5000 and frequency = 30 q.p. that

$$\mu = 2500\,\iota^{-50^\circ} \quad \text{(nearly)}.$$

Now
$$G = \frac{4\pi}{\Sigma \frac{L}{A\mu}}$$

and
$$\Sigma \frac{L}{A\mu} = \frac{1}{100}\left\{\frac{40}{2500}\iota^{50^\circ} + \cdot 2\right\}$$
$$= \frac{1}{10^5}\left\{16\iota^{50^\circ} + 200\right\}$$
$$= \frac{2106}{10^4}\iota^{3^\circ 18'}.$$

Hence
$$G = 5968\,\iota^{-3^\circ 18'}.$$

18. Returning to the alternator, if n be the number of armature turns, ν the number of field turns, and

$$x = \mathrm{a}_1 + \mathrm{a}_3 + \mathrm{a}_5 + \&c., \ \xi = \frac{\alpha_0}{2} + \alpha_2 + \alpha_4 + \&c.,$$

the armature and field currents respectively, the magnetizing current-turns M_x producing flux across the air-gap and through the armature in a direction axial to its windings are given by

$$M_x = nx + \nu\xi \cos \omega t,$$

and the current-turns M_y producing flux across the air-gap and through the armature in a direction parallel to the planes of the windings, and *behind* that of M_x by 90°, are given by

$$M_y = \nu\xi \sin \omega t,$$

or in vector notation (see § 4)

$$M_x = \Sigma\left[n\mathrm{a}_q + \frac{\nu}{2}(\alpha_{q-1} + \alpha_{q+1})_q\right]$$

$$M_y = \frac{\nu}{2}\iota^{-\frac{\pi}{2}}\Sigma(\alpha_{q-1} - \alpha_{q+1})_q,$$

which produce the armature fluxes A_x and A_y given by

$$A_x = \Sigma G_q\left[n\mathbf{a}_q + \frac{\nu}{2}(\alpha_{q-1} + \alpha_{q+1})_q\right]$$

$$A_y = \frac{\nu}{2}\iota^{-\frac{\pi}{2}}\Sigma G_q(\alpha_{q-1} - \alpha_{q+1})_q,$$

where $G_q = g_q\iota^{-\delta q}$ and q any odd number.

[Note that the directions of A_x and A_y are fixed in the armature.]

Now if magnetic leakage be otherwise taken account of, the flux in the stator must be continuous with that in the rotor so that the flux F looped on the field-windings at any instant is given by

$$F = A_x \cos \omega t + A_y \sin \omega t,$$

which by means of the relations in § 4 can be reduced to

$$F = \Sigma\left[\nu G_p\alpha_p + \frac{n}{2}(G_{p-1}\mathbf{a}_{p-1} + G_{p+1}\mathbf{a}_{p+1})_p\right]$$

where p is even and $2G_p = G_{p-1} + G_{p+1}$.

19. If l' be the self-inductance in the armature circuit either external to the armature or due to magnetic leakage in it, and if λ' be a similar quantity for the field circuit, the equations for the two circuits are

$$rx + l'\frac{dx}{dt} + n\frac{d}{dt}A_x = 0$$

$$\rho\xi + \lambda'\frac{d\xi}{dt} + \nu\frac{d}{dt}F = \eta$$

where r, ρ, η, x, ξ, have the same significations as in § 1.

Substituting in these equations from § 18, and then equating separately to zero each set of vector terms of the same order, we obtain the two series of vector equations,

$$r\mathbf{a}_q + q\omega l'\iota^{\frac{\pi}{2}}\mathbf{a}_q + nq\omega\iota^{\frac{\pi}{2}}G_q\left\{n\mathbf{a}_q + \frac{\nu}{2}(\alpha_{q-1} + \alpha_{q+1})\right\} = 0$$

$$\rho\alpha_p + p\omega\lambda'\iota^{\frac{\pi}{2}}\alpha_p + \nu p\omega\iota^{\frac{\pi}{2}}\left\{\nu G_p\alpha_p + \frac{n}{2}(G_{p-1}\mathbf{a}_{p-1} + G_{p+1}\mathbf{a}_{p+1})\right\} = 0$$

with $\alpha_0 = \dfrac{2\eta}{\rho}$, where q is odd and p even.

These reduce at once to the two series,

$$\alpha_{q-1}+t_q \mathrm{G}_q \mathbf{a}_q+\alpha_{q+1}=0$$

$$\mathrm{G}_{p-1}\mathbf{a}_{p-1}+\tau_p\alpha_p+\mathrm{G}_{p+1}\mathbf{a}_{p+1}=0$$

or, after putting $\mathbf{a}'_q$ for $\mathrm{G}_q\mathbf{a}_q$, to

$$\alpha_{q-1}+t_q\mathbf{a}'_q+\alpha_{q+1}=0$$

$$\mathbf{a}'_{p-1}+\tau_p\alpha_p+\mathbf{a}'_{p+1}=0$$

equations of exactly the same form as those for the simple case but in which the t and τ operators are now given by

$$t_q=\frac{2}{n\nu\mathrm{G}_q^2}\left\{n^2\mathrm{G}_q+l'-\frac{r}{q\omega}\iota^{\frac{\pi}{2}}\right\}$$

$$\tau_p=\frac{2}{n\nu}\left\{\nu^2\mathrm{G}_p+\lambda'-\frac{\rho}{p\omega}\iota^{\frac{\pi}{2}}\right\}.$$

These operators having been calculated from known data, the solution for $\mathbf{a}'_1$, $\mathbf{a}'_3$, $\mathbf{a}'_5$, &c., α_2, α_4, &c., proceeds exactly as in the simple case, and as $\mathbf{a}'_1=\mathrm{G}_1\mathbf{a}_1$, $\mathbf{a}'_3=\mathrm{G}_3\mathbf{a}_3$, &c., $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$, &c., can then be obtained.

20. In § 16 it was shown that the iron loss (*i.e.* energy dissipated per sec. in the iron) in a magnetic circuit is

$$\tfrac{1}{2}\omega\Sigma q\frac{f_q^2}{g_q}\sin\delta_q\,;$$

hence the loss due to the flux A_x § 18 is

$$\tfrac{1}{2}\omega\Sigma q g_q\sin\delta_q\overline{n\mathbf{a}_q+\frac{\nu}{2}(\alpha_{q-1}+\alpha_{q+1})}^2$$

and that due to the flux A_y is

$$\tfrac{1}{2}\omega\Sigma q g_q\sin\delta_q\frac{\nu^2}{4}\overline{(\alpha_{q-1}-\alpha_{q+1})}^2$$

Adding these we find that the total iron loss in the generator is

$$\tfrac{1}{2}\omega\Sigma q g_q\sin\delta_q\left\{\overline{n\mathbf{a}_q+\frac{\nu}{2}(\alpha_{q-1}+\alpha_{q+1})}^2+\frac{\nu^2}{4}\overline{(\alpha_{q-1}-\alpha_{q+1})}^2\right\}.$$

Expanding and remembering that

$$\overline{\alpha+\beta}^2=\bar{\alpha}^2+\bar{\beta}^2+2\bar{\alpha}\bar{\beta}\cos\widehat{\alpha\beta},$$

and that

$$\alpha_{q-1}+\alpha_{q+1}=-t_q\mathrm{G}_q\mathbf{a}_q \qquad \text{(see § 19)}$$

we find that the total iron loss is equal to

$$\Sigma\left[\tfrac{1}{2}q\omega g_q \sin\delta_q \left\{\frac{\nu^2}{2}(\bar{\alpha}^2_{q-1}+\bar{\alpha}^2_{q+1})-n^2\bar{\mathbf{a}}^2_q\right\}-\left\{r\sin^2\delta_q+q\omega l'\sin\delta_q\cos\delta_q\right\}\bar{\mathbf{a}}^2_q\right]$$

where q is odd.

21. An approximate determination of the effect of iron loss on the performance of an alternator can be obtained by taking all the G operators for its magnetic circuit as equal to G_1, that is, equal to the one for the fundamental harmonic of the armature flux.

Making this simplification in the equations of § 19, we find that $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$, &c., α_0, α_2, &c., are connected by the two series of equations

$$\alpha_{q-1}+t_q\mathbf{a}_q+\alpha_{q+1}=0$$

$$\mathbf{a}_{p-1}+\tau_p\alpha_p+\mathbf{a}_{p+1}=0$$

with $\alpha_0=2\eta/\rho$, in which

$$t_q=\frac{2}{n\nu G}\left\{n^2G+l'-\frac{r}{q\omega}\iota^{\frac{\pi}{2}}\right\}=\frac{2}{n\nu}\left\{n^2+\frac{l'}{g}\iota^{\delta}-\frac{r}{q\omega g}\iota^{\frac{\pi}{2}+\delta}\right\}$$

$$\tau_p=\frac{2}{n\nu G}\left\{\nu^2G+\lambda'-\frac{\rho}{p\omega}\iota^{\frac{\pi}{2}}\right\}=\frac{2}{n\nu}\left\{\nu^2+\frac{\lambda'}{g}\iota^{\delta}-\frac{\rho}{p\omega g}\iota^{\frac{\pi}{2}+\delta}\right\}$$

as $G=g\iota^{-\delta}$.

Putting l for gn^2, λ for $g\nu^2$, m for $gn\nu$, and—remembering that δ is a small angle (see § 17)—unity for $\cos\delta$, we find that

$$t_q=D_q\iota^{-f_q},\quad \tau_p=\Delta_p\iota^{-\phi_p},$$

where

$$D_q^2=\frac{4}{m^2}\left\{(l+l')^2+\frac{r^2}{q^2\omega^2}+2\frac{lr}{q\omega}\sin\delta\right\}$$

$$D_q\sin f_q=\frac{2r}{q\omega m}-\frac{2l'}{m}\sin\delta$$

$$\Delta_p^2=\frac{4}{m^2}\left\{(\lambda+\lambda')^2+\frac{\rho^2}{p^2\omega^2}+2\frac{\lambda\rho}{p\omega}\sin\delta\right\}$$

$$\Delta_p\sin\phi_p=\frac{2\rho}{p\omega m}-\frac{2\lambda'}{m}\sin\delta.$$

The t and τ operators having been calculated from these formulæ, the rest of the solution for this case follows in every particular the course for the simple case fully explained in §§ 8, 9.

22. In order to illustrate the practical application of the foregoing theory, I will determine the performance of a small two-pole alternator when carrying a rather heavy non-inductive load.

The details of the alternator are as follows :—

$$\text{Armature}\begin{cases}\text{diameter } 12 \text{ cm.}\\ \text{length} \quad 8 \text{ cm.}\\ \text{turns } n=100.\\ \text{resistance } \cdot 25 \text{ ohm.}\end{cases}$$

Air-gap $=1$ mm.

$$\text{Field}\begin{cases}\text{turns } \nu=400.\\ \text{resistance } \rho=3 \text{ ohms.}\\ \text{exciter, three storage-cells ; } \tau=6\cdot 6 \text{ volts.}\end{cases}$$

Frequency $100/\pi$, *i. e.* $\omega=200$.
Magnetic leakage $=5$ per cent.
Flux operator $G=5000\,\iota^{-3^\circ}$.

Let the external resistance in the armature circuit in the case in hand be 4·75 ohms so that $r=5$ ohms.

Hence (see § 21)

$$l = 5\,.\,10^7, \qquad l' = \cdot 05\, l = \cdot 25\,.\,10^7$$
$$\lambda = 8\,.\,10^8, \qquad \lambda' = \cdot 05\lambda = 4\,.\,10^7$$
$$m = 2\,.\,10^8, \qquad r = 5\,.\,10^9, \qquad \rho = 3\,.\,10^9$$
$$\omega = 200, \qquad \delta = 3^\circ \qquad \text{and}$$
$$\bar{\alpha}_0 = 2\eta/\rho = \cdot 44 \text{ (absolute).}$$

Using these values for the constants and the formulæ in § 21 we obtain the t and τ operators which are given in the following table :—

$t_q=D_q\iota^{-f_q}$.			$\tau_p=\Delta_p\iota^{-\phi_p}$.		
q.	D_q.	f_q.	p.	Δ_p.	ϕ_p.
1	·592	24° 51′	2	8·4	0° 22′
3	·535	8 48	4	8·4	8′
5	·530	5 18	6	8·4	1′
7	·528	3 45	8	8·4	−1′
9	·527	2 54	10	8·4	−2′
11	·526	2 21	12	8·4	−3′
13	·526	1 57	14	8·4	−4′
15	·526	1 40	16	8·4	−5′
17	·526	1 28	18	8·4	−5′

[Note that the τ operators are all practically equal and simple numerical multipliers (see § 8).]

And from these, by the method explained in § 8 we obtain the S, Σ, continued fraction operators which are given in the following table:—

$S_q = s_{q\iota} - b_q$.			$\Sigma_p = \sigma_{p\iota} - \beta_p$.		
q.	s_q.	b_q.	p.	σ_p.	β_p.
1	·457	36° 32′	2	5·82	7° 46′
3	·366	15 32	4	5·63	5 8
5	·353	9 50	6	5·57	3 40
7	·350	7 9	8	5·55	2 54
9	·348	5 34	10	5·53	2 18
11	·346	4 30	12	5·51	1 51
13	·346	3 42	14	5·50	1 21

And from these as explained in § 9 we obtain the different harmonics of both the armature current and the induced field current. These also are given in tabular form:—

$x = \Sigma x_q \sin(q\omega t - \frac{\pi}{2} + c_q)$.			$\xi = \cdot 22 + \Sigma \xi_p \sin(p\omega t + \frac{\pi}{2} + \gamma_p)$.		
q.	x_q.	c_q.	p.	ξ_p.	γ_p.
1	·963	36° 32′	2	·166	44° 18′
3	·453	59 50	4	·080	64 58
5	·228	74 48	6	·041	78 28
7	·117	85 37	8	·021	88 31
9	·061	94 5	10	·011	96 23
11	·032	100 53	12	·006	102 44
13	·017	106 26	14	·003	107 47

The virtual armature current in amperes being equal to $10\sqrt{\frac{1}{2}\Sigma x_q^2}$ is 7·75 amps., and the terminal virtual voltage being 4·75 times this is 36·8 volts. The no-load voltage for the same exciting current is 62·2.

The virtual value of the alternating current induced in the field circuit in amperes being equal to $10\sqrt{\frac{1}{2}\Sigma \xi_p^2}$ is 1·34.

The copper losses are:—

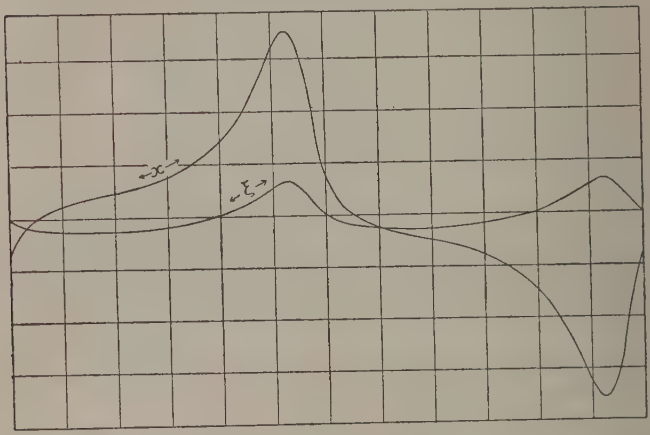
In the armature........................ 15 watts.
In the field due to exciting current... 12 "
In the field due to induced current... 5·4 "

The total iron loss calculated by means of the formula in § 20 is 20 watts.

The total losses are therefore 52·4 watts, and as the output is 285 watts the efficiency is 84 per cent.

In fig. 3 are plotted the wave forms of both the armature

Fig. 3.—Theoretical armature current (x), and induced field-current (ξ), in a fully loaded alternator.



and induced field-currents determined above, correctly as regards both their relative amplitudes and phases.

23. In designing the field of an alternator attention should be given to the fact that the conductor has to carry not only the exciting current, but also the induced field-current, which, as we have seen, may at full load attain a relatively large value. In addition it should not be forgotten that in the field-magnet cores there is the associated alternating flux which causes some additional heat.

It is well known that in a case of excessive heating in the field, reduction of the heating is effected by the employment of heavy closed copper conductors, called dampers, embracing the field-magnet poles.

To explain this action, let us consider a two-pole machine on each field pole of which is a damper.

Neglecting magnetic leakage and iron loss, if ζ be the current in each damper, the magnetic flux through the armature windings is

$$g\{nx+(\nu\xi+2\zeta)\cos\omega t\},$$

and that through the field windings and the dampers is

$$g\{\nu\xi+2\zeta+nx\cos\omega t\},$$

so that the equations connecting x, ξ, and ζ are

$$\left.\begin{aligned} &rx+gn\frac{d}{dt}\{nx+(\nu\xi+2\zeta)\cos\omega t\}=0,\\ &\rho\xi+g\nu\frac{d}{dt}\{\nu\xi+2\zeta+nx\cos\omega t\}=\eta,\\ &z\zeta+g\ \frac{d}{dt}\{\nu\xi+2\zeta+nx\cos\omega t\}=0,\end{aligned}\right\}\quad . \quad . \quad (\text{VII.})$$

where z is the resistance of each damper and the other symbols have the same significations as in the previous sections of this paper.

Obviously there is no constant term in ζ, and considering only the variable terms (harmonics) in ξ, we see at once that

$$\nu z\zeta=\rho\xi,$$

from which it follows that

$$\nu z+2\zeta=\nu\xi(1+\kappa)=\nu\xi' \text{ say},$$

where

$$\kappa=\frac{2\rho}{\nu^2 z},$$

and that

$$2z\zeta^2=\kappa\rho\xi^2.$$

The first two of equations VII. may now be written

$$rx+gn\frac{d}{dt}\{nx+\nu\xi'\cos\omega t\}=0,$$

$$\frac{\rho}{1+\kappa}\xi'+g\nu\frac{d}{dt}\{\nu\xi'+nx\cos\omega t\}=\frac{\eta}{1+\kappa},$$

η being divided by $1+\kappa$ as the constant term in ξ' is equal to the constant term in ξ (ζ having no constant term).

Now x and ξ' determined from these equations will be

very approximately the same as x and ξ determined from the equations in § 1 for the alternator without dampers; for α_0 the given vector is the same for both, as are also all the t operators.

The τ operators differ in that for ρ, $\rho/(1+\kappa)$ is substituted, but in § 8 and in note § 22 it is shown that the τ operators for generators as ordinarily constructed are practically independent of the value of ρ the field resistance.

Hence we see that ξ' is the alternating field current if the dampers are absent, ξ its value when the dampers are attached and these currents are connected by the relation

$$\xi' = (1+\kappa)\xi.$$

In addition as $2z\zeta^2 = \kappa\rho\xi^2$,

$$\rho\xi^2 = \frac{2}{\kappa}z\zeta^2 = \frac{1}{(1+\kappa)^2}\rho\xi'^2.$$

Hence if H′ be the copper loss in the field coils due to induced alternating current when the generator is without dampers, H the same when dampers are attached, and h the loss in the dampers when attached,

$$\mathrm{H} = \frac{\mathrm{H}'}{(1+\kappa)^2}, \quad h = \frac{\kappa \mathrm{H}'}{(1+\kappa)^2}, \quad \mathrm{H}+h = \frac{\mathrm{H}'}{1+\kappa}.$$

If we assume that the mean length of a field turn is equal to the length of a damper turn, it is easy to show that κ is the ratio of the volume of copper in the dampers to the volume of copper in the field windings when there is no resistance external to the windings in the field circuit. If there is resistance external to the windings in the field circuit, κ is greater than the above volume ratio.

The magnetic flux in the field coils being equal to

$$g\{\nu\xi' + nx\cos\omega t\}$$

is practically unaffected by the presence of dampers, so that the iron loss in the field-magnets remains the same.

24. If a source of alternating E.M.F. = E where

$$\mathrm{E} = \mathrm{E}_1 \sin(\omega t + h_1) + \mathrm{E}_3 \sin(3\omega t + h_3) + \&\text{c.},$$

$$= e_1 + e_3 + e_5 + \&\text{c. (vectors)}$$

be included in the armature circuit, and if the armature

rotate in synchronism with this E.M.F., we have the case of the synchronous A.C. motor.

In this case the armature and field currents x and ξ are connected by the equations (see § 1)

$$rx + \frac{d}{dt}(lx + m\xi \cos \omega t) = \Sigma \mathrm{E}_q \sin (q\omega t \mp h_q)$$

$$\rho\xi + \frac{d}{dt}(\lambda\xi + mx \cos \omega t) = \eta.$$

Assuming as in § 2 that

$$x = \mathbf{a}_1 + \mathbf{a}_3 + \mathbf{a}_5 + \&c.,$$

$$\xi = \frac{\alpha_0}{2} + \alpha_2 + \alpha_4 + \&c.,$$

and proceeding exactly as in § 5, we obtain the infinite series of equations

$$\begin{aligned}
t_1\mathbf{a}_1 + \alpha_2 \qquad\qquad\qquad\qquad &= -\alpha_0 - \kappa_1\\
\mathbf{a}_1 + \tau_2\alpha_2 + \mathbf{a}_3 \qquad\qquad\quad &= 0\\
\alpha_2 + t_3\mathbf{a}_3 + \alpha_4 \qquad\quad &= -\kappa_3\\
\mathbf{a}_3 + \tau_4\alpha_4 + \mathbf{a}_5 \quad &= 0\\
\alpha_4 + t_5\mathbf{a}_5 + \alpha_6 &= -\kappa_5\\
\&c., \ \&c.,&
\end{aligned}$$

in which

α_0 = the vector to the point $2\eta/\rho$, $\pi/2$, as before

and $$\kappa_q = \frac{2}{qm\omega}\iota^{\frac{\pi}{2}}e_q,$$

where Σe_q is the applied E.M.F., and the t and τ operators have the same values as in § 5.

Solving for $\mathbf{a}_1$ we find that

$$\mathrm{P}_1\mathbf{a}_1 = -\Pi_2(\alpha_0 + \kappa_1)_1 - \Pi_4(\kappa_3)_1 - \Pi_6(\kappa_5)_1 - \&c.,$$

where P_1, Π_2, Π_4, &c., are the infinite determinant operators whose leading terms are t_1, τ_2, τ_4, τ_6, &c., respectively as in § 6.

Reducing to the continued fraction operators of § 7 we obtain

$$\mathbf{a}_1 = -\frac{1}{\mathrm{S}_1}(\alpha_0 + \kappa_1)_1 - \frac{1}{\mathrm{S}_1\Sigma_2\mathrm{S}_3}(\kappa_3)_1 - \frac{1}{\mathrm{S}_1\Sigma_2\mathrm{S}_3\Sigma_4\mathrm{S}_5}(\kappa_5)_1 - \&c.,$$

and using the equations

$$\alpha_2 = -t_1\mathrm{a}_1 - \alpha_0 - \kappa_1\,;\ \mathrm{a}_3 = -\tau_2\alpha_2 - \mathrm{a}_1\,;\ \text{\&c.},$$

the successive harmonics of the armature and field currents can be obtained.

25. If in the last example the E.M.F. inserted in the armature circuit be sinusoidal and equal to $\mathrm{E}\sin(\omega t + h) = e$ (a vector), the solution will (see equations in last paragraph) obviously be identical with that for the simple generator given in §§ 5 *et seq.*, when in the latter $\alpha_0 + \kappa$ is substituted for α_0 where

$$\kappa = \frac{2}{\omega m}\iota^{\frac{\pi}{2}}e,$$

and in this case it is important to know the condition which determines whether the machine will run as a motor and develop mechanical power.

In § 12 the *driving* torque T was shown to be given by

$$\mathrm{T} = -\frac{m}{4}\{\mathrm{S}\alpha_0\mathrm{a}_1 + \mathrm{Sa}_1\alpha_2 + \mathrm{S}\alpha_2\mathrm{a}_3 + \mathrm{Sa}_3\alpha_4 + \text{\&c.}\}$$

and T must be negative for a motor.

Now, as

$$\mathrm{a}_1 = -\frac{1}{\mathrm{S}_1}(\alpha_0 + \kappa)_1 = -\frac{\iota^{\mathrm{b}_1}}{s_1}(\alpha_0 + \kappa)_1 \text{ (see § 8)}$$

$$\mathrm{S}\alpha_0\mathrm{a}_1 = -\frac{1}{s_1}\left\{\mathrm{S}\alpha_0\,.\,\iota^{\mathrm{b}_1}\alpha_0 + \frac{2}{\omega m}\mathrm{S}\alpha_0\,.\,\iota^{\frac{\pi}{2}+\mathrm{b}_1}e\right\}$$

$$= -\frac{1}{s_1}\left\{\bar{\alpha}_0^2\sin\mathrm{b}_1 + \frac{2}{\omega m}\bar{\alpha}_0\bar{e}\sin(\mathrm{b}_1 + h)\right\}.$$

Again, as

$$\alpha_2 = -\frac{1}{\Sigma_2}\mathrm{a}_1 = -\frac{\iota^{\beta_2}}{\sigma_2}\mathrm{a}_1,$$

$$\mathrm{Sa}_1\alpha_2 = -\frac{\sin\beta_2}{\sigma_2}\bar{\mathrm{a}}_1^2 = -\frac{\sin\beta_2}{s_1^2\sigma_2}\overline{(\alpha_0 + \kappa)}^2\,;$$

similarly

$$\mathrm{S}\alpha_2\mathrm{a}_3 = -\frac{\sin\mathrm{b}_3}{s_1^2\sigma_2^2 s_3}\overline{(\alpha_0 + \kappa)}^2,\ \text{\&c., \&c.}\,;$$

but

$$\overline{(\alpha_0+\kappa)}^2 = \bar{\alpha}_0^2+\bar{\kappa}^2+2\bar{\alpha}_0\bar{\kappa}\cos\widehat{\alpha_0\kappa}$$

$$= \bar{\alpha}_0^2+\frac{4}{\omega^2m^2}\bar{e}^2+\frac{4}{\omega m}\bar{\alpha}_0\bar{e}\cos h\ ;$$

and if

$$\text{B} = \frac{\sin\beta_2}{s_1^2\sigma_2}+\frac{\sin \mathbf{b}_3}{s_1^2\sigma_2^2s_3}+\frac{\sin\beta_4}{s_1^2\sigma_2^2s_3^2\sigma_4}+\&\text{c.},$$

we find that

$$\frac{4\text{T}}{m}=\left\{\frac{\sin\mathbf{b}_1}{s_1}+\text{B}\right\}\bar{\alpha}_0^2+\frac{2}{\omega m}\left\{\frac{\sin(\mathbf{b}_1+h)}{s_1}+2\text{B}\cos h\right\}\bar{\alpha}_0\bar{e}+\frac{4}{\omega^2m^2}\text{B}\bar{e}^2,$$

which must be negative if the machine runs as a motor.

Now, s_1, σ_2, s_3, σ_4, &c., $\sin\mathbf{b}_1$, $\sin\beta_2$, $\sin\mathbf{b}_3$, are all essentially positive and therefore B is so. Also $\bar{\alpha}_0$ and $\bar{e}$ are essentially positive. So, in order that the machine may work as a motor, h, the phase angle of the applied E.M.F. must have such a value as to make the above expression for T negative.

The power supplied by the source e being the mean value of the product of e and x, that is of e and $\mathbf{a}_1$, that is of e and $-\frac{1}{s_1}\iota^{\mathbf{b}_1}(\alpha_0+\kappa)$ is

$$= -\frac{1}{2s_1}\bar{\alpha}_0\bar{e}\sin(h-\mathbf{b}_1)+\frac{1}{2s_1}\bar{e}\bar{\kappa}\sin\mathbf{b}_1 \text{ (see § 11)}$$

$$= -\frac{\bar{e}}{2s_1}\left\{\bar{\alpha}_0\sin(h-\mathbf{b}_1)-\frac{2}{\omega m}\bar{e}\sin\mathbf{b}_1\right\}.$$

It is interesting to note that the armature and alternating-field currents which flow when an E.M.F. $=\text{E}\sin(\omega t+h)$ is acting in the armature circuit, the angular velocity of the armature is ω, and the exciting current C, would be unchanged if the E.M.F. $\text{E}\sin(\omega t+h)$ be removed, the speed maintained, and the exciting field-current changed from

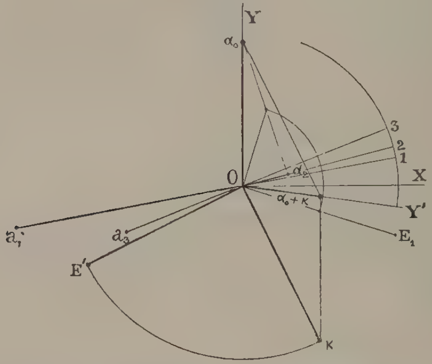
$$\text{C to }\sqrt{\text{C}^2+\frac{2}{\omega m}\text{CE}\cos h+\frac{1}{\omega^2m^2}\text{E}^2}.$$

This follows immediately from the vector equations in § 31 connecting $\mathbf{a}_1$, α_2, $\mathbf{a}_3$, &c., with $\alpha_0+\kappa$ when κ_3, κ_5, κ_7, &c., are zero.

26. The case of the synchronous motor with sinusoidal applied E.M.F., discussed in the last paragraph, can easily be represented geometrically.

In fig. 4, let $O\alpha_0$, taken in the axis of Y be equal (as in § 13) to twice the steady exciting current of the machine. Draw

Fig. 4.



the vector OE′ to represent in amplitude and phase $2/\omega m$ times the applied E.M.F.; that is, if the latter

$$= e' \sin(\omega t + h), \qquad OE' = 2e'/\omega m,$$

and the angle from OX to OE′ measured in the positive direction is $= h$.

Rotating OE′ forward through 90° gives us κ of § 25, and completing the parallelogram $\alpha_0 O \kappa$, its diagonal is $\alpha_0 + \kappa$ in the line OY′.

Knowing the motor circuits, we can determine s_1, $\mathbf{b}_1$, σ_2, β_2, s_3, $\mathbf{b}_3$, &c., and then construct for $\mathbf{a}_1$, α_2, $\mathbf{a}_3$, α_4, &c., exactly as in § 13, except that in this construction the vector $\alpha_0 + \kappa$ takes the place of α_0 in § 13 (see § 24).

Now the mechanical torque *developed by* the machine is (see § 12)

$$= \frac{m}{4}\{S\alpha_0\mathbf{a}_1 + S\mathbf{a}_1\alpha_2 + S\alpha_2\mathbf{a}_3 + \&c.\}$$

$= \frac{m}{2}$ into the sum of the areas, attending to signs, of the triangles $\alpha_0 O \mathbf{a}_1$, $\mathbf{a}_1 O \alpha_2$, $\alpha_2 O \mathbf{a}_3$, &c.

But the triangles $a_1O\alpha_2$, α_2Oa_3, $a_3O\alpha_4$, &c., are all essentially negative [their sum is $= -\frac{1}{2}\overline{B\alpha_0 + \kappa^2}$ of § 25], so that if the machine is to develop mechanical power and run as a motor, the phase of OE′ must be such that the area of the triangle α_0Oa_1 is positive (as it is in fig. 4) and numerically greater than the sum of $a_1O\alpha_2$, α_2Oa_3, &c.

The power supplied by the source is

$$= \frac{1}{2} \cdot \frac{\omega m}{2} \overline{OE'} \cdot \bar{a}_1 \cos a_1OE',$$

$$= \frac{\omega m}{4} \bar{a}_1 \bar{\kappa} \sin a_1O\kappa,$$

$$= \frac{\omega m}{2} \times \text{area of triangle } a_1O\kappa,$$

and the power developed by the motor $= \omega T = \frac{\omega m}{2} \times$ sum of areas of the triangles α_0Oa_1, $a_1O\alpha_2$, α_2Oa_3, &c., attending to signs.

Hence the efficiency is equal to

$$\frac{\alpha_0Oa_1 + a_1O\alpha_2 + \alpha_2Oa_3 + \&c.}{a_1O\kappa}.$$

By rotating the vector from O to the middle point of $\alpha_0\alpha_2$ backwards through 90° and doubling we obtain OE_1, which represents in amplitude and phase $2/m\omega$ times the first harmonic of the total E.M.F. of the motor (see § 13).

This vector can now be compared with OE′ which represents $2/m\omega$ times the applied E.M.F.

Fig. 4 easily explains how, by increasing the exciting current of an A.C. motor, the phase of the armature current is advanced relative to that of the applied E.M.F.

Discussion.

Mr. W. Duddell expressed his interest in Prof. Lyle's paper, and remarked that the results which he had obtained were in accord with experiments made by himself and Dr. Marchant some years ago.

Dr. Russell congratulated the Author on having obtained such instructive solutions of the differential equations which determine the value of the armature and field currents in a simple generator. The subject of armature reaction has been carefully studied by electrical

engineers, and the literature of the subject is quite extensive. He referred in particular to the study made by Professor Blondel in 1900 of the ripples in the exciting current of a two-phase and a three-phase alternator. It was well known that in a single-phase machine the frequency of the ripple superposed on the exciting current by the alternate magnetising and demagnetising effect of the armature current was double the frequency of the armature current. This ripple disappears at no load, but in practice the ripples at no load are often very marked owing to pulsations of the reluctance due to slots in the armature. The latter ripples are much less pronounced at full load as the load circuit acts like a damping-coil; but new ripples due to the armature reaction appear, causing a distortion of the wave-form. As a rule, electrical engineers assumed the existence of a sine wave of armature current and then investigated the ampere-turns to be added or subtracted from the field coils so as to neutralise the magnetising or demagnetising effect produced. In this connexion Blondel's two-reaction method was extensively used as it enabled approximate values to be rapidly obtained. The effect of the reaction on the wave-form of the machine, however, had been practically neglected, and Professor Lyle deserves great credit for his solution. In connexion with the parallel running of turbo-alternators it was important, and the theory deserved careful study by engineers. The speaker thought that "dampers" were mainly used to prevent phase-swinging. He agreed with Professor B. Hopkinson that a similar effect might often be more economically produced by putting more copper on the field windings. The difficulties that arose in connexion with perfecting the theory of synchronous motors arose mainly from the difficulty of taking hysteresis into account in the mathematical equations. He thought that Professor Lyle's work in this direction was most valuable.

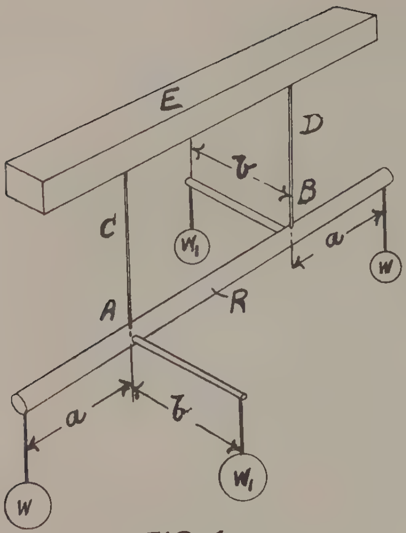


FIG. 1.

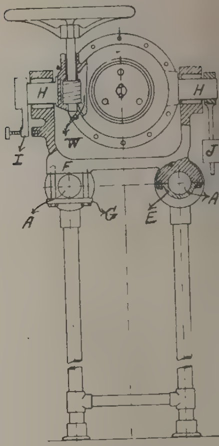
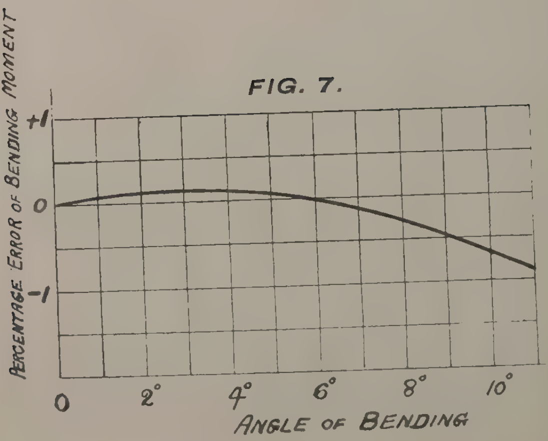


FIG 3

FIG. 7.



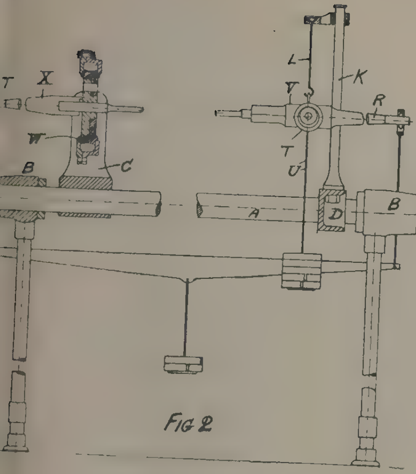


FIG 2

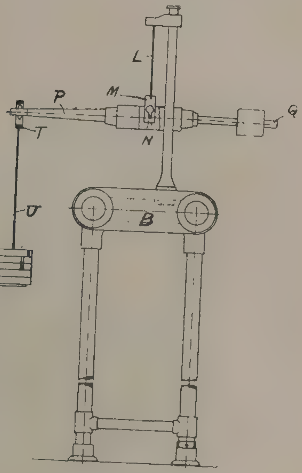


FIG 4

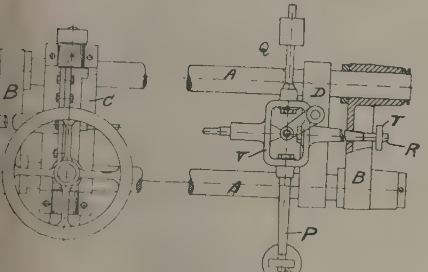


FIG 5

FIG 6

FIG. 9.

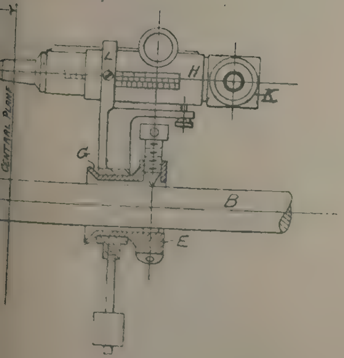


FIG. 10.

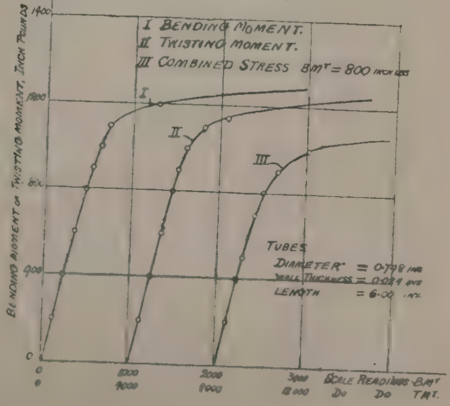


FIG. 8.

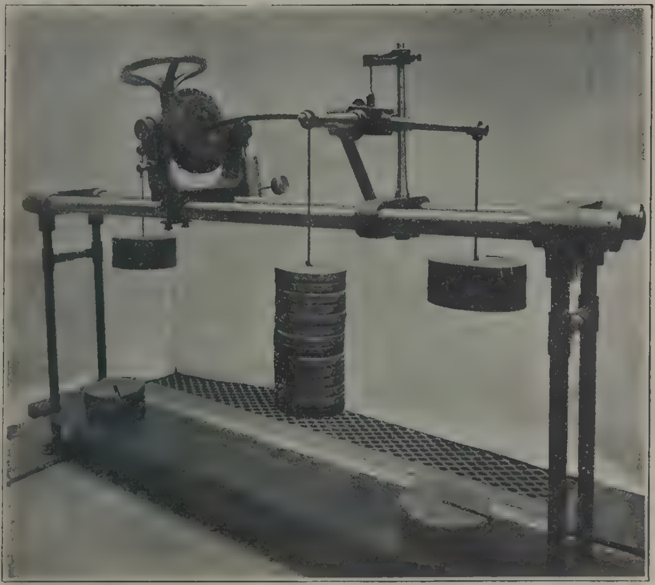


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Fig. 1.

$\mathcal{B}$-$\mathcal{H}$ Curves for Bars of circular section.

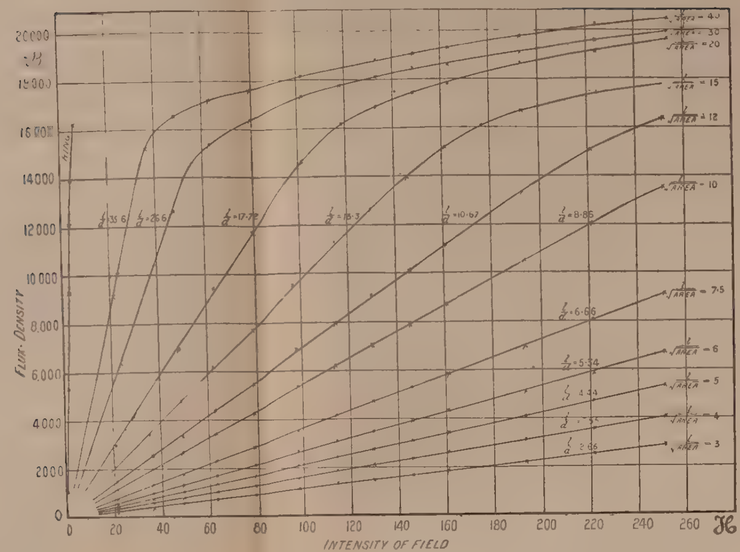


Fig. 2.

Coefficients of Demagnetization for Bars of circular section.

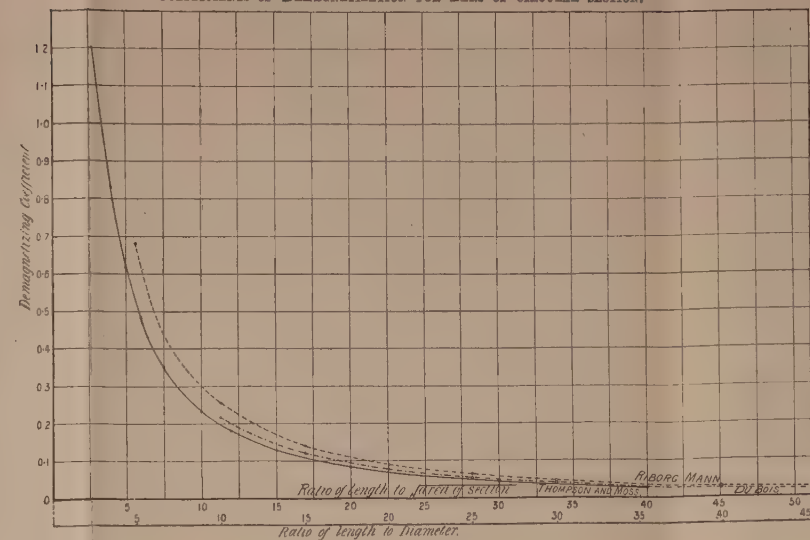


Fig. 3.

$\mathcal{B}$-$\mathcal{H}$ Curves for Bars of various forms of section. Area of section 1 sq. centimetre. Length 100 millimetres.

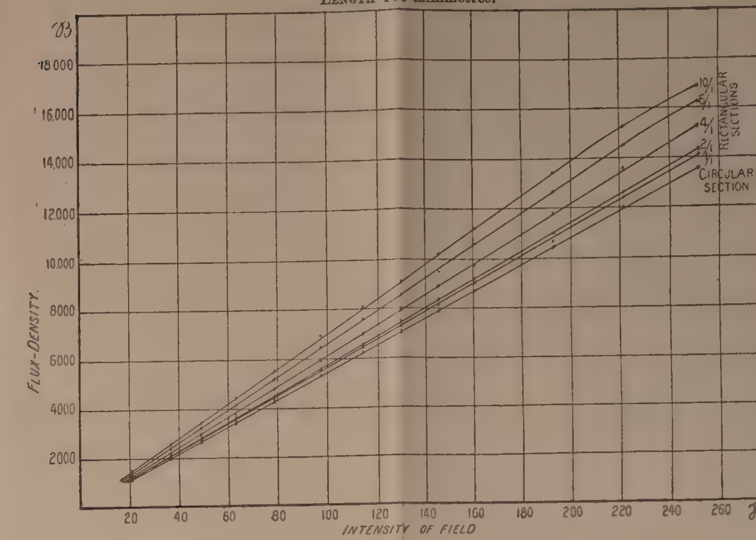


Fig. 4.

$\mathcal{B}$-$\mathcal{H}$ Curves for Bars of various forms of section. Area of section 1 sq. centimetre. Length 80 millimetres.

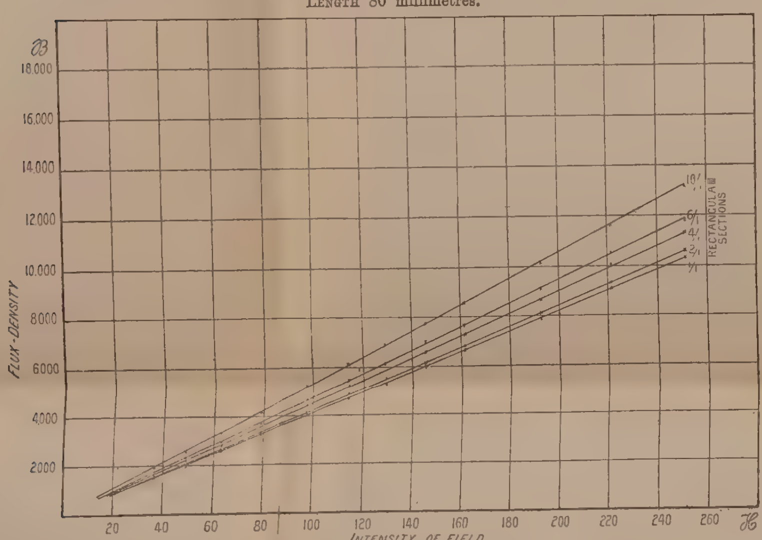


Fig. 5.

$\mathcal{B}$-$\mathcal{H}$ Curves for Bars of various forms of section. Area of section 1 sq. centimetre. Length 60 millimetres.

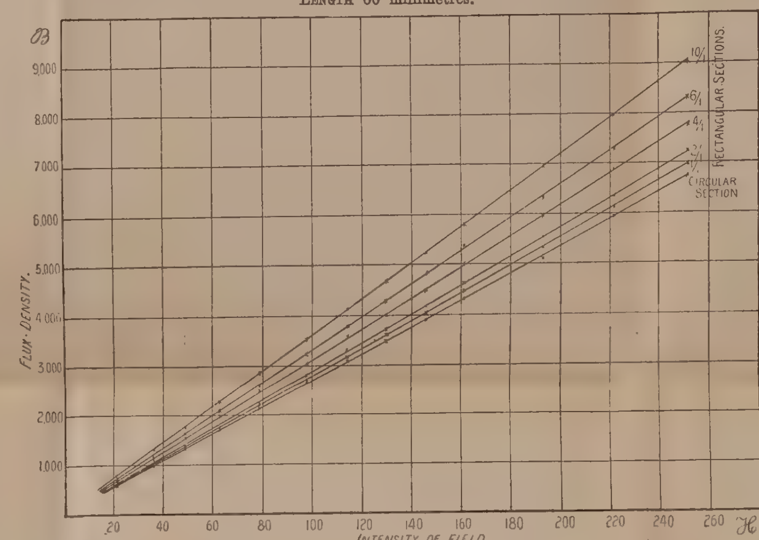


Fig. 6.

$\mathcal{B}$-$\mathcal{H}$ Curves for Bars of various forms of section. Area of section 1 sq. centimetre. Length 50 millimetres.

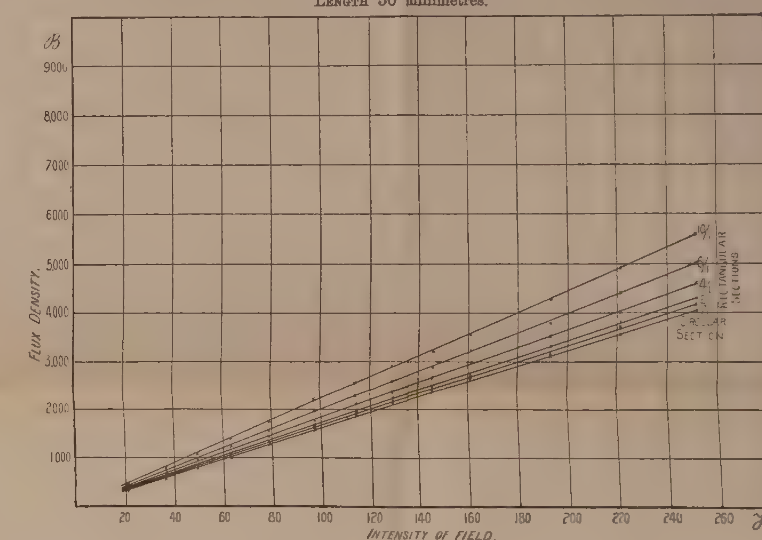


Fig. 7.

$\mathcal{B}$-$\mathcal{H}$ Curves for Bars of various forms of section. Area of section 1 sq. centimetre. Length 40 milimetres.

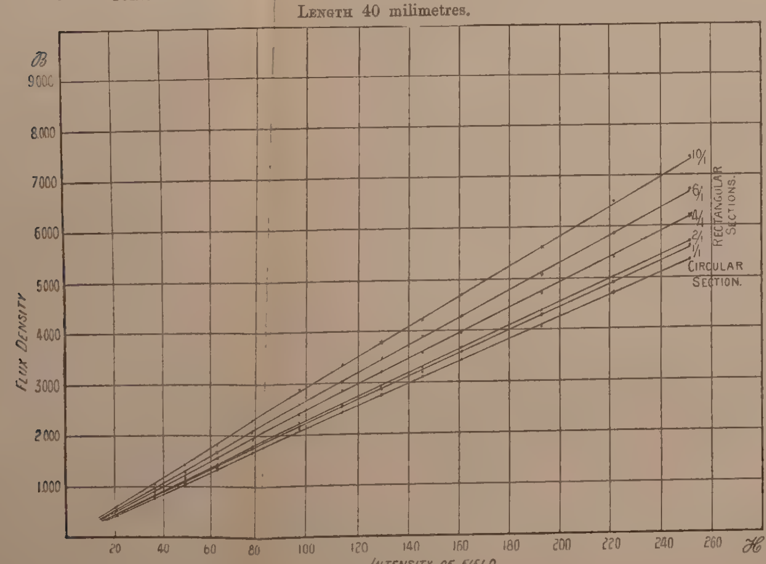


Fig. 8.

$\mathcal{B}$-$\mathcal{H}$ Curves for Bars of various forms of section. Area of section 1 sq. centimetre. Length 30 millimetres.

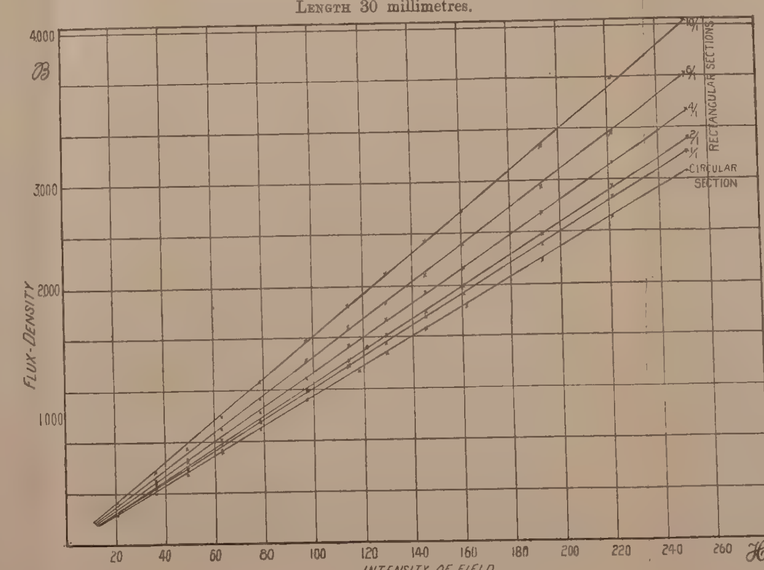
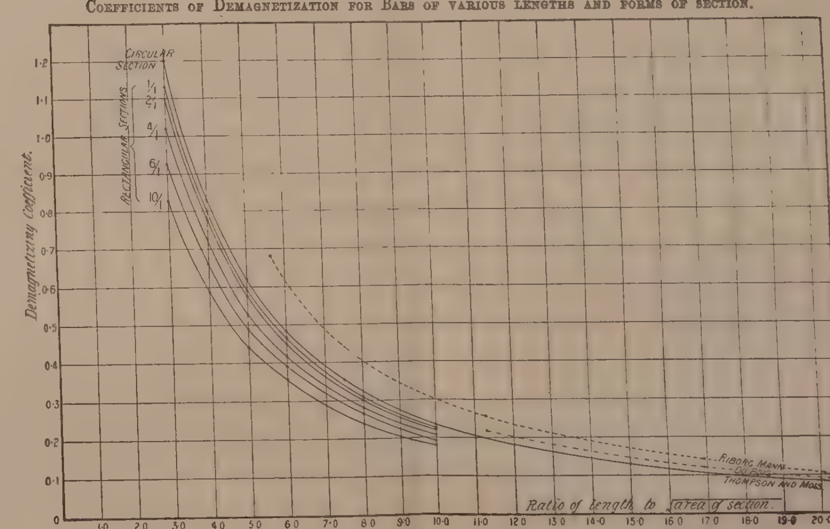
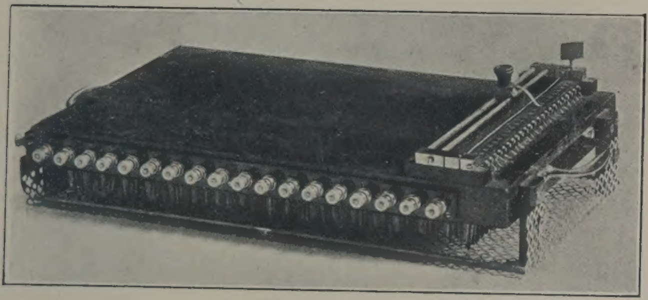
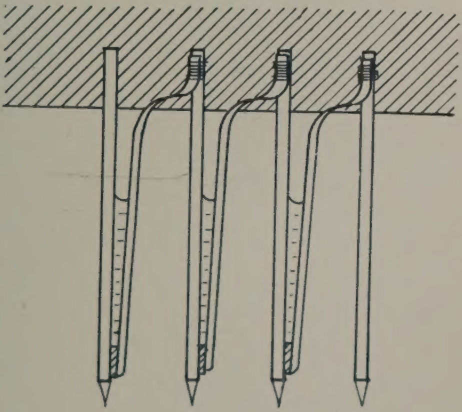


Fig. 9.

Coefficients of Demagnetization for Bars of various lengths and forms of section.







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